SHEAF THEORY IN SYMPLECTIC GEOMETRY

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Abstract. These are some notes on microlocal sheaf theory and its applications in symplectic and contact geometry. For the general theory of microlocal sheaves, the main source will be Kashiwara and Schapira's *Sheaves on Manifolds* and Schapira's *A Short Review to Microlocal Sheaf Theory*. For sheaf quantization of Legendrian isotopies the sources are Guillermou, Kashiwara and Schapira's *Sheaf Quantization of Hamiltonian Isotopies and Applications to Nondisplaceability Porblems* and Shende, Treumann and Zaslow's *Legendrian Knots and Constructible Sheaves*. For quantization of Legendrian submanifolds the source is Guillermou's *Quantization of Conical Lagrangian Submanifolds of Cotangent Bundles*. Other references include Viterbo's *An Introduction to Symplectic Topology through Sheaf Theory*, Tarmarkin's *Microlocal Conditions for Nondisplaceability*, Shende's online lecture notes and Nadler's online lecture notes.

Contents

1. General Sheaf Theory

1.1. **Sheaves On Manifolds.** Let M be a C^∞-manifold. We study sheaves on M with coefficient $\Bbbk$ where $\Bbbk$ is a field. They form an abelian category $Sh(M)$, and we can consider the unbounded derived category $DSh(M)$ and bounded derived category $D^bSh(M)$.

Suppose $f : M \to N$ is a continuous map. We recall some definitions. Let $\mathscr{G} \in Sh(N)$. The pull back $f^{-1}\mathscr{G}$ is defined by

$$\Gamma(U, f^{-1}\mathscr{G}) = \varinjlim_{V \supset f(U)} \Gamma(V, \mathscr{G}).$$

This is an exact functor, so we denote the derived functor again by f^{-1}. Let $\mathscr{F} \in Sh(M)$. The push forward $f_*\mathscr{F}$ is defined by

$$\Gamma(V, f_*\mathscr{F}) = \Gamma(f^{-1}(V), \mathscr{F}).$$

This is only left exact, and the derived functor will be denoted by Rf_*. Also recall that f_* is the left adjoint of f^{-1}.

Besides the push forward, we also have the proper push forward $f_!\mathscr{F}$ defined by

$$\Gamma(V, f_!\mathscr{F}) = \{s \in \Gamma(f^{-1}(V), \mathscr{F}) | \mathrm{supp} s \text{ compact}\}.$$

Again this is left exact, and the derived functor will be denoted by $Rf_!$.

One intuitive understanding of Rf_* and $Rf_!$ is that the push forward gives you standard cohomology, and the proper push forward gives you compactly supported cohomology. When $f : M \to \mathrm{pt}$, it's not hard to show

$$H^*(Rf_*\Bbbk_M) \simeq H^*(M; \Bbbk), \;\; H^*(Rf_!\Bbbk_M) \simeq H^*_c(M; \Bbbk).$$

Indeed as in cohomology theory we have change-of-variable formula for integration, here we also have a base change formula for proper push-forward.

Proposition 1.1 (Base change formula)**.** *Suppose $f \circ g' = g \circ f'$. Then $g^{-1} \circ Rf_! \simeq Rf'_! \circ (g')^{-1}$.*

Proof. We only prove the proposition in the underived setting. First one can build a canonical morphism $f_! \circ g'_* \to g_* \circ f'_!$ (we leave it to the readers as an exercise). By adjunction this will induce a morphism $g^{-1} \circ f_! \to f'_! \circ (g')^{-1}$. It is an isomorphism because

$$\begin{aligned}(g^{-1} \circ f_!\mathscr{F})_x = (f_!\mathscr{F})_{g(x)} &= \Gamma_c(f^{-1}(g(x)), \mathscr{F}) \\ &\simeq \Gamma_c((f')^{-1}(x), (g')^{-1}\mathscr{F}) \simeq (f'_!(g')^{-1}\mathscr{F})_x.\end{aligned}$$

Since $g' : (f')^{-1}(x) \to f^{-1}(g(x))$ is a homeomorphism. ☐

A natural question is, is there a right adjoint functor of the proper push forward? Here is the answer.

Definition 1.1. *Let $f : M \to N$ be continuous. $f_!$ has finite cohomological dimensions. Let $\mathscr{G} \in D^bSh(N)$. Then the complex of sheaves $f^!\mathscr{G}$ is defined by*

$$R\Gamma(U, f^!\mathscr{G}) = RHom(f_!K_U, \mathscr{G}),$$

where $K_U \to \Bbbk_U \to 0$ is a c-soft resolution.

The functor $f^!$ is the right adjoint of $f_!$. Now we're able to generalize Poincare duality to Verdier duality on sheaves. This requires the notion of a dualizing sheaf (or a dualizing complex).

Definition 1.2. *Let $f : M \to N$ be continuous. Then the dualizing sheaf is*

$$\omega_{M/N} = f^! \Bbbk_N.$$

In particular, write $\omega_M = \omega_{M/\mathrm{pt}}$. Let $\mathscr{F} \in D^b Sh(M)$. Then the Verdier dual of $\mathscr{F}$ is

$$D_M \mathscr{F} = R\mathscr{H}om(\mathscr{F}, \omega_M).$$

The dual of $\mathscr{F}$ is

$$D'_M \mathscr{F} = R\mathscr{H}om(\mathscr{F}, \Bbbk_M).$$

Let $f : M \to N$ be a submersion with fiber dimension l. Then the orientation sheaf is defined by

$$or_{M/N} = H^{-l}\omega_{M/N}.$$

In fact, when $N = \mathrm{pt}$ and M is orientable, $or_M = \Bbbk_M$. In general,

$$\Gamma(U, or_M) = Hom(H^n_c(U; \Bbbk), \Bbbk).$$

Here are some basic properties of the proper pullback functor. We first remind the readers of two other functors that will be used frequently in the future. For $\mathscr{F} \in Sh(M)$, $Z \subset M$ a locally closed subset,

$$\mathscr{F}_Z = i_{Z,*} i_Z^{-1} \mathscr{F} = \mathscr{F} \otimes \Bbbk_Z, \;\; \Gamma_Z(\mathscr{F}) = Hom(\Bbbk_Z, \mathscr{F}).$$

There are corresponding derived functors in $D^b Sh(M)$.

Proposition 1.2. *Let $Z \subset M$ be a closed subset, $U = M \backslash Z$ and $i : U \to M$. Then there are exact triangles*

$$R\Gamma_Z(\mathscr{F}) \to \mathscr{F} \to Ri_* i^{-1} \mathscr{F} \xrightarrow{[1]}, \quad Ri_! i^{-1} \mathscr{F} \to \mathscr{F} \to \mathscr{F}_Z.$$

Proposition 1.3. *Let $f : M \to N$ be a homeomorphism onto a locally closed subset $f(M) \subset N$. Then*

$$f^! \simeq f^{-1} \circ R\Gamma_{f(M)}.$$

Proof. When f is an embedding we know that $f^{-1} f_! \simeq \mathrm{id}$.

$$\begin{aligned} RHom(\mathscr{F}, f^! \mathscr{G}) \simeq & RHom(f_! \mathscr{F}, \mathscr{G}) \simeq RHom(f_! \mathscr{F} \otimes \Bbbk_{f(M)}, \mathscr{G}) \\ \simeq & RHom(f_! \mathscr{F}, R\Gamma_{f(M)}(\mathscr{G})) \simeq RHom(\mathscr{F}, f^{-1} R\Gamma_{f(M)}(\mathscr{G})). \end{aligned}$$

□

Proposition 1.4. *Let $f : M \to N$ and $\mathscr{F}, \mathscr{G} \in D^b Sh(N)$. Then*

$$f^! R\mathscr{H}om(\mathscr{F}, \mathscr{G}) \simeq R\mathscr{H}om(f^{-1}\mathscr{F}, f^! \mathscr{G}).$$

Proof. Consider any sheaf $\mathscr{H} \in D^b Sh(M)$. Then

$$\begin{aligned} R\mathscr{H}om(\mathscr{H}, f^! R\mathscr{H}om(\mathscr{F}, \mathscr{G})) \simeq & R\mathscr{H}om(f_! \mathscr{H}, R\mathscr{H}om(\mathscr{F}, \mathscr{G})) \\ \simeq & R\mathscr{H}om(f_! \mathscr{H} \otimes^L \mathscr{F}, \mathscr{G}) \\ \simeq & R\mathscr{H}om(f_!(\mathscr{H} \otimes^L f^{-1}\mathscr{F}), \mathscr{G}) \\ \simeq & R\mathscr{H}om(\mathscr{H}, R\mathscr{H}om(f^{-1}\mathscr{F}, f^! \mathscr{G})). \end{aligned}$$

The second last isomorphism is because one can choose a flat resolution of $\mathscr{F}$ and when $\mathscr{F}$ is flat

$$\begin{aligned} \big(f_!(\mathscr{H} \otimes f^{-1}\mathscr{F})\big)_x \simeq & \Gamma_c(f^{-1}(x), \mathscr{H} \otimes f^{-1}\mathscr{F}) \simeq \Gamma_c(f^{-1}(x), \mathscr{H} \otimes (\mathscr{F}_x)_M) \\ \simeq & \Gamma_c(f^{-1}(x), \mathscr{H}) \otimes \mathscr{F}_x = (f_! \mathscr{H} \otimes \mathscr{F})_x\,. \end{aligned}$$

□

The following proposition gives an alternative description of $R\mathscr{H}om(\mathscr{F},\mathscr{G})$ in terms of the diagnol in $M \times M$. The result will be used later.

Proposition 1.5. *Let $\mathscr{F},\mathscr{G} \in D^bSh(M)$, and $\Delta \subset M \times M$ the diagnol. Then*

$$R\mathscr{H}om(\mathscr{F},\mathscr{G}) \simeq R\Gamma_\Delta(R\mathscr{H}om(p_1^{-1}\mathscr{F}, p_2^!\mathscr{G}))|_\Delta.$$

Proof. Denote by $\delta : \Delta \to M \times M$ the embedding. Then

$$\begin{aligned}\delta^{-1}R\Gamma_\Delta(R\mathscr{H}om(p_1^{-1}\mathscr{F}, p_2^!\mathscr{G})) \simeq& \delta^! R\mathscr{H}om(p_1^{-1}\mathscr{F}, p_2^{-1}\mathscr{G})\\ \simeq& R\mathscr{H}om(\delta^{-1}p_1^{-1}\mathscr{F}, \delta^! p_2^!\mathscr{G})\\ \simeq& R\mathscr{H}om(\mathscr{F},\mathscr{G}).\end{aligned}$$

□

1.2. **Singular Support.** We know that a sheaf $\mathscr{F}$ is defined by its corresponding local sections $U \mapsto \Gamma(U,\mathscr{F})$. When a sheaf is locally constant, this gives us a local system (which is equivalent to a vector bundle with a flat connection). However, this is not always the case.

Suppose we want to measure locally how a sheaf fails to be a constant sheaf, then a natural way would be to measure where parallel transport fails to transport points on the stalk. Therefore we need the following information: (1). a point; (2). a tangent vector along which we are doing parallel transport. Unfortunately we don't have the notion of parallel transport on a general sheaf, so instead we have to look at local sections. Namely we measure where local sections fail to extend.

Definition 1.3. *Given M a manifold and $\mathscr{F} \in D^bSh(M)$, the singular support is a set $SS(\mathscr{F}) \subset T^*M$ so that $(x,\xi) \in SS(\mathscr{F})$ if there is $\varphi \in C^\infty(M)$, $\varphi(x) = 0, d\varphi(x) = \xi$ so that*

$$R\Gamma_{\varphi^{-1}([0,\infty))}(\mathscr{F})_x \neq 0.$$

Proposition 1.6. *(1). $SS(\mathscr{F}) \cap X = supp(\mathscr{F})$; (2). $SS(\mathscr{F}[1]) = SS(\mathscr{F})$;*

(3). Let $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \xrightarrow{[1]}$ be an exact triangle. Then for $i \neq j \neq k$,

$$SS(\mathscr{F}_i) \subset SS(\mathscr{F}_j) \cup SS(\mathscr{F}_k),$$
$$SS(\mathscr{F}_i)\backslash SS(\mathscr{F}_j) \cup SS(\mathscr{F}_j)\backslash SS(\mathscr{F}_i) \subset SS(\mathscr{F}_k).$$

The definition of NOT being in the singular support is independent of the choice of the function $\varphi \in C^\infty(M)$. In fact this can be proved using the non-characteristic deformation lemma, though not so obvious.

Basically, the lemma says that as long as we deform the level set $\varphi^{-1}(0)$ without crossing the singular support, the cohomology group won't change, which tells us that indeed the notion of singular support is detecting if sections of sheaves can propagate/extend in certain directions.

Theorem 1.7 (Non-characteristic deformation lemma)**.** *Let $\{U_t\}_{t\in\mathbb{R}}$ be a family of open subsets in M, $\mathscr{F} \in D^b(M)$ with compact support, so that*

(1). $U_t = \bigcup_{s<t} U_s$, $\forall\, t \in \mathbb{R}$;

(2). For $Z_s = \overline{\bigcap_{t>s} U_t \backslash U_s}$, $R\Gamma_{M\backslash U_t}(\mathscr{F})_x = 0$ as long as $x \in Z_s\backslash U_t$.

Then for all $t \in \mathbb{R}$, we have

$$R\Gamma\Big(\bigcup_{s\in\mathbb{R}} U_s, \mathscr{F}\Big) \xrightarrow{\sim} R\Gamma\left(U_t, \mathscr{F}\right).$$

The following lemma shows that the sheaf category $D^b(M)$ is invariant under homotopy equivalence. The proof relies on the non-characteristic deformation lemma.

Lemma 1.8. *Let $f : M \to N$ be a homotopy equivalence. Then $Rf_* f^{-1}\mathscr{F} \xrightarrow{\sim} \mathscr{F}$.*

Proof. Let $h : M \times [0,1] \to M$ be a homotopy so that $h_0 = \mathrm{id}$. First we prove that $Rh_{1,*}h_1^{-1} = \mathrm{id}$. Let $p : M \times [0,1] \to M$ be the projection. Then

$$Rh_{i,*} \circ h_i^{-1} = Rh_* \circ Rj_{i,*} \circ j_i^{-1} \circ h^{-1}, \;\; i = 0, 1.$$

It suffices to show that $Rj_{0,*} \circ j_0^{-1} = Rj_{1,*} \circ j_1^{-1}$. Since $p \circ j_i = \mathrm{id}$, there exist natural transformations

$$Rj_{i,*} \circ j_i^{-1} \to Rj_{i,*} \circ Rp_*, \;\; i = 0, 1.$$

Showing this is an isomorphism, by adjunction, is equivalent to showing for a constant sheaf $\mathscr{F} = M_{[0,1]}$, $R\Gamma([0,1], \mathscr{F}) \to \mathscr{F}_t$ is always an isomorphism. This follows from theorem 1.7. □

The following theorem, known as the microlocal cut-off lemma, is important when estimating the singular support of a sheaf.

Definition 1.4. *Let E be a vector space. $s : E \times E \to E$, $(v_1, v_2) \mapsto v_1 + v_2$. Let $\mathscr{F}, \mathscr{G} \in D^b(E)$. Then the convolution functor is*

$$\mathscr{F} \star \mathscr{G} = Rs_*(p_1^{-1}\mathscr{F} \otimes p_2^{-1}\mathscr{G}).$$

In the following theorem, by a conical subset γ in a vector space E we mean a subset that is invariant under scaling. For a conical subset $\gamma \subset E$, its polar set is

$$\gamma^\vee = \{v \in E^\vee \,|\, \langle u, v\rangle \geq 0\}.$$

Theorem 1.9 (Microlocal Cut-off Lemma)**.** *Let E be a vector space, $\gamma \subset E$ be a closed cone and $\mathscr{F} \in D^b(E)$. Let*

$$\mathscr{F}' = \mathrm{Cone}(\Bbbk_\gamma \star \mathscr{F} \to \Bbbk_0 \star \mathscr{F}).$$

Then $SS(\mathscr{F}') \cap (E \times (\gamma^\vee)^\circ) = \emptyset$, and $\mathscr{F}' \simeq 0$ iff

$$SS(\mathscr{F}) \subset E \times (\gamma^\vee)^\circ.$$

Note that Kashiwara-Schapira used the push-forward functor via $(-\gamma)$-topology instead of convolution with $\Bbbk_\gamma$, but the results are the same. In fact, one can prove that

$$R\Gamma(U, \Bbbk_\gamma \star \mathscr{F}) \simeq R\Gamma(U - \gamma, \mathscr{F}).$$

Proof of Theorem 1.9. Without loss of generality, we always assume that $\mathscr{F}$ has compact support. By corollary of non-characteristic deformation lemma we have

$$\begin{aligned} R\Gamma(U, \Bbbk_\gamma \star \mathscr{F}) &\simeq R\Gamma(s^{-1}(U), \pi_1^{-1}\Bbbk_\gamma \otimes \pi_2^{-1}\mathscr{F}) \simeq R\Gamma(s^{-1}(U) \cap (\gamma \times E), \pi_2^{-1}\mathscr{F}) \\ &\simeq R\Gamma(\pi_2(s^{-1}(U) \cap (\gamma \times E)), \mathscr{F}) \simeq R\Gamma(U - \gamma, \mathscr{F}). \end{aligned}$$

First assume that $\Bbbk_\gamma \star \mathscr{F} \simeq \Bbbk_0 \star \mathscr{F}$. Then for any $\xi \notin \gamma^\vee$, choose $\varphi(x) = \langle x, \xi\rangle$. Then $\varphi^{-1}((-\infty, 0)) - \gamma = E$.

$$R\Gamma(\varphi^{-1}((-\infty, 0)), \mathscr{F}) \simeq R\Gamma(\varphi^{-1}((-\infty, 0)) - \gamma, \mathscr{F}) \simeq R\Gamma(E, \mathscr{F}).$$

Let $i : \varphi^{-1}((-\infty, 0)) \to E$, then by taking derived global sections of the exact triangle

$$R\Gamma_{\varphi^{-1}([0,+\infty))}(\mathscr{F}) \to \mathscr{F} \to i_*i^{-1}\mathscr{F} \xrightarrow{[1]},$$

one can get that $R\Gamma_{\varphi^{-1}([0,+\infty))}(\mathscr{F}) \simeq 0$, which means $SS(\mathscr{F}) \subset E \times \gamma^\vee$.

Then assume that $SS(\mathscr{F}) \subset E \times \gamma^\vee$. We show that for any open ball U we have an isomorphism $R\Gamma(U - \gamma, \mathscr{F}) \xrightarrow{\sim} R\Gamma(U, \mathscr{F})$, which will show, since open balls form a neighbourhood system on a manifold, that

$$\Bbbk_\gamma \star \mathscr{F} \xrightarrow{\sim} \Bbbk_0 \star \mathscr{F}.$$

In fact we appeal to the non-characteristic deformation lemma 1.7. Choose $U_t\,(t\in\mathbb{R})$ with smooth boundary so that $\bigcup_{t\in\mathbb{R}} U_t = U-\gamma, \bigcap_{t\in\mathbb{R}} U_t = U$, and the outward conormal directions of U_t is contained in $E^\vee\backslash\gamma^\vee$. Then the condition on $SS(\mathscr{F})$ ensures that

$$R\Gamma_{E\backslash U_t}(\mathscr{F})_x \simeq 0,\ \forall\, x\in Z_t\backslash U_t.$$

Hence by non-characteristic deformation lemma, we know that

$$R\Gamma(U-\gamma,\mathscr{F}) \xrightarrow{\sim} R\Gamma(U,\mathscr{F}),$$

which means $\mathscr{F}'\simeq 0$. □

Here are some examples of singular supports of sheaves.

Let $U\subset M$ be an open submanifold with smooth boundary ∂U. Then $SS(\Bbbk_U) = (T^*_{\partial U}M)_-$, $SS(\Bbbk_{\overline{U}}) = (T^*_{\partial U}M)_+$, where $+$ stands for the outward pointing normal vectors and $-$ stands for the inward pointing ones. We only check the first one here. Suppose U is locally defined by a coordinate function $\varphi_1>0$. Then on the coordinate chart, write $U=\varphi^{-1}((0,+\infty))$,

$$\begin{aligned} R\Gamma_{\overline{U}}(\Bbbk_U)_0 &= R\mathscr{H}om(\Bbbk_{\overline{U}},\Bbbk_U)_0 = RHom_{D^b(\mathbb{R}^n)}(\Bbbk_{\overline{U}},\Bbbk_U)\\ &= RHom_{D^b(\mathbb{R}^n)}\left(i_{U,*}i_U^{-1}\Bbbk, i_{U,!}i_U^{-1}\Bbbk\right) = H^*_c(U;\Bbbk) = \Bbbk[n]. \end{aligned}$$

Along all the other directions, it is easy to check that $R\Gamma_{\varphi^{-1}([0,+\infty))}(\Bbbk_{\varphi_1>0})_0 = 0$. Therefore $SS(\Bbbk_U) = (T^*_{\partial U}M)_-$.

Let $N\subset M$ be a closed submanifold. Then $SS(\Bbbk_N) = T^*_N M$. In fact suppose that locally N is defined by coordinate functions $\varphi_1 = ... = \varphi_k = 0$. Then on the coordinate chart, write $N = 0\times\mathbb{R}^{n-k}$, $N^\perp = \mathbb{R}^k\times 0$, $U_i = \varphi_i^{-1}((0,+\infty))$, $V_i = U_i\cap N^\perp$,

$$\begin{aligned} R\Gamma_{\overline{U_i}}(\Bbbk_N)_0 &= R\mathscr{H}om(\Bbbk_{\overline{U_i}},\Bbbk_N)_0 = RHom_{D^b(\mathbb{R}^n)}(\Bbbk_{\overline{U_i}},\Bbbk_N)\\ &= RHom_{D^b(\mathbb{R}^n)}(\pi_{N^\perp}^{-1}\Bbbk_{\overline{V_i}}, \pi_{N^\perp}^{-1}\Bbbk_0) = RHom_{D^b(N^\perp)}(\Bbbk_{\overline{V_i}},\Bbbk_0).\\ &= RHom_{D^b(N^\perp)}(i_{\overline{V_i},!}i_{\overline{V_i}}^{-1}\Bbbk,\Bbbk_0) = RHom_{D^b(N^\perp)}(\Bbbk, i_{\overline{V_i},*}i^!_{\overline{V_i}}\Bbbk_0)\\ &= R\Gamma(N^\perp,\Bbbk_0\otimes\Bbbk) = \Bbbk,\ \ 1\le i\le k. \end{aligned}$$

Therefore $T^*_N M\subset SS(\Bbbk_N)$. On the other hand, let $\varphi_{k+1},...,\varphi_n$ coordinate functions on N. On the coordinate chart,

$$R\Gamma_{\varphi_i^{-1}([0,+\infty))}(\Bbbk_{\varphi_1=...=\varphi_k=0})_0 = 0,\ \ k+1\le i\le n.$$

This shows that in fact $SS(\Bbbk_N) = T^*_N M$.

Finally, let $U = \{(x,y)| -x^{3/2} < y \le x^{3/2}\}\subset\mathbb{R}^2$. Then

$$SS(\Bbbk_U) = \{(x,y,\xi,\eta)|x = -(2\xi/3\eta)^3, y = (2\xi/3\eta)^2\}.$$

In fact, set $\varphi_\pm(x,y) = y\pm x^{3/2}$ when $x\ge 0$ and y when $x\le 0$. Then

$$SS(\Bbbk_{\varphi_\pm^{-1}([0,+\infty))}) = \{(x,y,\lambda d\varphi_\pm(x,y))|\lambda\varphi_\pm(x,y) = 0, \lambda\ge 0, \varphi_\pm\ge 0\}.$$

By proposition 1.6, we know that the result stated above is true. This is a standard local model of a cusp for front projections of Legendrian knots.

1.3. **Functorial Properties.**

In this section we study how the singular support changes under functors between sheaves.

Proposition 1.10. *Let $f: M\to N$ be a submersion, $\mathscr{F}\in D^b(N)$. Then*

$$SS(f^{-1}\mathscr{F}) = f_d f_\pi^{-1}(SS(\mathscr{F})).$$

Proof. First we prove that
$$SS(f^{-1}\mathscr{F}) \subset f_d f_\pi^{-1}(SS(\mathscr{F})).$$
Choose a local chart so that locally $M = \mathbb{R}^n$, $N = \mathbb{R}^k$, and $f : M \to N$ is the projection $\pi : (x', x'') \mapsto x'$. Pick $(x', x''; \xi', \xi'') \notin f_d(f_\pi^{-1}(SS(\mathscr{F})))$. If $\xi'' \neq 0$, then we let $\varphi(x) = \langle x'', \xi''\rangle$. Since $f^{-1}\mathscr{F}$ is constant along v when $\langle v, \xi''\rangle < 0$.
$$R\Gamma_{\varphi \geq 0}(f^{-1}\mathscr{F})_x \simeq 0.$$
Thus $(x', x''; \xi', xi'') \notin SS(f^{-1}\mathscr{F})$. If $\xi'' = 0$, then actually $(x', \xi') \notin SS(\mathscr{F})$. Let $\varphi(x) = \langle x', \xi'\rangle$. Since
$$\Gamma_{\varphi\circ f \geq 0}(f^{-1}\mathscr{F})_x = \Gamma_{\varphi \geq 0}(\mathscr{F})_{x'},$$
$(x', x''; \xi', 0) \notin SS(f^{-1}\mathscr{F})$.

Then we show that
$$f_d f_\pi^{-1}(SS(\mathscr{G})) \subset SS(f^{-1}\mathscr{G}).$$
This is because for any $y \in f^{-1}(x)$, we have
$$R\Gamma_{\varphi \geq 0}(\mathscr{F})_x = R\Gamma_{\varphi\circ f \geq 0}(f^{-1}\mathscr{F})_y,$$
which is essentially the chain rule. □

Definition 1.5. *Let $S \subset M$ be a subset. Then $N_x S = T_x M \backslash C_x(M\backslash S, S)$, $N_x^* S = (N_x S)^\vee$, $NS = \bigcup_{x\in M} N_x S$ and $N^*S = \bigcup_{x\in M} N_x^* S$.*

Theorem 1.11. *(1). Let $i : U \to M$ be an open embedding.*

(i). Assume that $SS(\mathscr{F}) \cap N^*U^{op} \subset M \subset T^*M$. Then
$$SS(Ri_* i^{-1}\mathscr{F}) \subset SS(\mathscr{F}) + N^*U;$$
(ii). Assume that $SS(\mathscr{F}) \cap N^*U \subset M \subset T^*M$. Then
$$SS(Ri_! i^{-1}\mathscr{F}) \subset SS(\mathscr{F}) - N^*U;$$
(2). Let $Z \subset M$ be a closed subset.

(i). Assume that $SS(\mathscr{F}) \cap N^*Z \subset M \subset T^*M$. Then
$$SS(R\Gamma_Z(\mathscr{F})) \subset SS(\mathscr{F}) - N^*Z;$$
(ii). Assume that $SS(\mathscr{F}) \cap N^*Z^{op} \subset M \subset T^*M$. Then
$$SS(\mathscr{F}_Z) \subset SS(\mathscr{F}) + N^*Z.$$

Proof. (1). After choosing a local chart, we mat assume that M is a vector space. (i). For $\xi \notin (SS(\mathscr{F}) \cap \pi^{-1}(x)) + N_x^*U$, we show that $(x, \xi) \notin SS(Ri_* i^{-1}\mathscr{F})$. Now
$$(N_x^*U - \mathbb{R}_{\geq 0}\xi) \cap (-SS(\mathscr{F})) \subset M \subset T^*M.$$
Choose a conical neighbourhood $\gamma \subset T_x^*M$ of $(N_x^*U - \mathbb{R}_{\geq 0}\xi)$ that is disjoint from $-SS(\mathscr{F})$. Now consider $\gamma^\vee \subset M$. Choose a neighbourhood V of $x \in M$ such that $V \times \gamma^\vee \cap SS(\mathscr{F}) \subset M$. Let $U_0 \subset U_1 \subset U_0 \cup U$ be invariant under $\gamma^\vee$-translations. Then there are no differences between sections on U_0 and the ones on U_1,
$$(\Bbbk_{\gamma^\vee} \star R\Gamma_{M\backslash U_0}(\mathscr{F}))_{U_1} = 0.$$
Note that
$$\gamma^\vee \subset \{v \in M \,|\, \langle v, \xi\rangle < 0\} \cup \{0\}.$$
Hence U is invariant under $\gamma^\vee$-translation. Therefore
$$(\Bbbk_{\gamma^\vee} \star R\Gamma_{M\backslash U_0}(Ri_* i^{-1}\mathscr{F}))_{U_1} \simeq Ri_* i^{-1}(\Bbbk_\gamma \star R\Gamma_{M\backslash U_0}(\mathscr{F}))_{U_1} = 0.$$
This completes the proof of (i). For (ii) the proof is basically the same.

(2). Let $U = M\backslash Z$ and $i : U \to M$ be the embedding. The result follows from the exact triangles

$$R\Gamma_Z(\mathscr{F}) \to \mathscr{F} \to Ri_* i^{-1}\mathscr{F} \xrightarrow{[1]}, \ \ Ri_! i^{-1}\mathscr{F} \to \mathscr{F} \to \mathscr{F}_Z \xrightarrow{[1]} .$$

Thus we are through. □

Definition 1.6. *Let $f : M \to N$ be a continuous map between manifolds, $\Lambda \subset T^*N$ be a closed conical subset. f is non-characteristic for Λ (or Λ is non-characteristic for f) if*

$$f^*(\Lambda) \cap T^*_M N \subset M \subset f^* T^* N.$$

For $\mathscr{F} \in D^b(N)$, f is non-characteristic for $\mathscr{F}$ if it is for $SS(\mathscr{F})$.

Theorem 1.12. *Let $\mathscr{F} \in D^b(N)$ and $f : M \to N$ be non-characteristic. Then*

$$SS(f^{-1}\mathscr{F}) \subset f_d f_\pi^{-1}(SS(\mathscr{F})).$$

Proof. We decompose the map $f : M \to N$ into a closed embedding $\mathrm{id} \times f : M \to M \times N$ and a submersion $M \times N \to N$. Theorem 1.10 has already dealt with the submersion case, so it suffices to check the closed embedding case.

For closed embeddings $Z \to M$, by induction on dimensions, we may assume that Z is a hypersurface. In addition, we may assume that $M\backslash Z = U_+ \sqcup U_-$, and the corresponding embeddings are $i_\pm : U_\pm \to M$. Then consider the exact sequence

$$Ri_{+,!} i_+^{-1}\mathscr{F} \oplus Ri_{-,!} i_-^{-1}\mathscr{F} \to \mathscr{F} \to \mathscr{F}_Z \xrightarrow{x} .$$

Then by Theorem 1.11 we can tell that

$$SS(\mathscr{F}_Z) \subset SS(\mathscr{F}) + T^*_Z M.$$

This completes the proof. □

Proposition 1.13. *Let M, N be C^∞-manifolds, $\mathscr{F} \in D^b(M)$, and $\mathscr{G} \in D^b Sh(N)$. Then*

$$SS(\pi_M^{-1}\mathscr{F} \otimes \pi_N^{-1}\mathscr{G}) \subset SS(\mathscr{F}) \times SS(\mathscr{G}),$$

$$SS(R\mathscr{H}om(\pi_M^{-1}\mathscr{F}, \pi_N^{-1}\mathscr{G})) \subset (-SS(\mathscr{F})) \times SS(\mathscr{G}).$$

Definition 1.7. *Let $A \subset T^*M, B \subset T^*N$. Then*

$$C_\mu(A, B) = C_{T^*_M(M\times N)}(A, -B) \subset T_{T^*_M(M\times N)} T^*(M \times N) \simeq T^*(M \times_N TN).$$

Let $p : M \times_N TN \to M$ be the projection. Then

$$f^\#(A, B) = p_\pi p_d^{-1}(C_\mu(A, B)) = T^*M \cap C_\mu(A, B).$$

When $M = N$ and $f = \mathrm{id}$, then $A \hat{+} B = \mathrm{id}^\#(A, -B) \subset T^*M$.

Theorem 1.14. *Let $i : U \to M$ be an open embedding, $\mathscr{F} \in D^b(U)$. Then*

$$SS(Ri_*\mathscr{F}) \subset SS(\mathscr{F}) \hat{+} N^*U,$$

$$SS(Ri_!\mathscr{F}) \subset SS(\mathscr{F}) \hat{+} (-N^*U).$$

The idea of the proof is the following. We want to show that $\mathscr{F}$ is non-characteristic for N^*U. Then one can apply Theorem 1.11. However, this cannot be done because first there is not even a pull-back functor here, and second directly showing non-characteristicity seem to be hard. Therefore we approximate U by a sequence of subsets U_t and prove non-characteristicity for $j_t : U_t \to M$. In order to do that we need the following lemma.

Definition 1.8. *Let $\{A_n, \rho_{m,n}\}_{m,n\geq 0}$ be a projective system of abelian groups. It satisfies Mittag-Leffler condition if $\{\rho_{m,n}(X_n)\}_{n\geq m}$ is stationary for any $m \geq 0$.*

Lemma 1.15. *Let $\mathscr{F} \in D^b(M)$, $\{U_n\}_{n\geq 0}$ be an increasing sequence of open subsets and $\{Z_n\}_{n\geq 0}$ a decreasing sequence of closed subsets. Set $U = \bigcup_{n\geq 0} U_n$, $Z = \bigcap_{n\geq 0} Z_n$.*

(1). The natural map $H^j_Z(U, \mathscr{F}) \to \varprojlim_{n\geq 0} H^j_{Z_n}(U_n, \mathscr{F})$ is surjective;

(2). Assume that $\{H^{j-1}_{Z_n}(U_n, \mathscr{F})\}_{n\geq 0}$ satisfies Mittag-Leffler condition, then $H^j_Z(U, \mathscr{F}) \to \varprojlim_{n\geq 0} H^j_{Z_n}(U_n, \mathscr{F})$ is an isomorphism.

Proof of Theorem 1.14. We only show the first assertion. Let $(x_0, \xi_0) \notin \underline{SS(\mathscr{F})}\hat{+}N^*U$, and in fact assume that $x_0 \in \mathrm{supp}(\mathscr{F}) \cap \partial U$, $\xi_0 \neq 0$. This means $(x_0, \xi_0) \notin \overline{SS(\mathscr{F})}$. Since $\xi_0 \notin N^*_{x_0}U$, we may assume that for some closed cone γ we have $U + \gamma = U$, $N^*_{x_0}U \subset (\gamma^\vee)^\circ \cup \{0\}$ and $\xi_0 \notin \gamma^\vee$.

We choose $v_0 \in \gamma^\circ$ and define in a local chart

$$H_s = \{x \in \mathbb{R}^n |\, \langle x - x_0, \xi_0\rangle > -s\},\;\; U_{s,t} = \{x \in \mathbb{R}^n | x - tv_0(\langle x - x_0, \xi_0\rangle + s) \in U\}.$$

They satisfy the following conditions that (for $0 < t < t'$)

$$\overline{U}_{s,t} \cap H_s \subset \overline{U}_{s,t'} \cap H_s \subset U,\;\; \bigcup_{t>0} U_{s,t} \cap H_s = U \cap H_s.$$

Now we claim that one can find a neighbourhood $V \times W$ of (x_0, ξ_0) such that for $V_s = V \cap H_s$, $s, t > 0$ small,

$$SS(\mathscr{F}) \cap (-N^*U_{s,t}) \cap \pi^{-1}(V_s) \subset M \subset T^*M;$$
$$(SS(\mathscr{F}) + N^*U_{s,t}) \cap (V_s \times W) = \emptyset.$$

If either of the assertion fails, then one can choose sequences such that

$$s_n, t_n \to 0,\;\; x_n \to x_0,\;\; \zeta_n \in N^*_{x_n}U_{s_n,t_n}\backslash\{0\},$$
$$(x_n, \xi_n) \in SS(\mathscr{F}),\;\; \xi_n + \zeta_n = c\tilde{\xi}_n,\;\; \tilde{\xi}_n \to \xi_0$$

($c = 0$ for the first condition, $c = 1$ for the second). We define (y_n, η_n) by

$$y_n = x_n + t_n v_0(\langle x_n - x_0, \xi_0\rangle + s_n),\;\; \xi_n = \eta_n - t_n \langle \eta_n, v_0\rangle\, \xi_0,$$

and set $\rho_n = \zeta_n + \eta_n$. Then $\eta_n \in \gamma^\vee$ and

$$\xi_0, \tilde{\xi}_n, \eta_n, \rho_n \in -\{v_0\}^\vee,\;\; |\eta_n| \leq C\langle \eta_n, v_0\rangle,\;\; C|\rho_n| \geq -\langle \rho_n, v_0\rangle.$$

Therefore $\rho_n/|\rho_n| \to \xi_0/|\xi_0|, (x_n, \zeta_n/|\rho_n|) \in SS(\mathscr{F}), (y_n, \eta_n/|\rho_n|) \in N^*U$. Hence $(x_0, \xi_0/|\xi_0|) \in SS(\mathscr{F})\hat{+}N^*U$. A contradiction. This proves the assertions.

Fix $s > 0$ small. Let $\mathscr{F}_t = Rj_{s,t,*}(\mathscr{F}|_{U_{s,t}})$. Then the non-characteristicity tells us that

$$SS(\mathscr{F}_t) \cap (V_s \times W) = \emptyset.$$

Let $U_0 \subset U_1$ be invariant under γ'-translation for $\gamma' \subset \{\xi_0\}^\vee \cup \{0\}$ and $x_0 \in U_1\backslash U_0 \subset H_s$. Then

$$(\Bbbk_{\gamma'} \star R\Gamma_{U_1\backslash U_0}(\mathscr{F}_t)) = 0,\;\; \forall\, t > 0.$$

Now we apply Proposition 1.15 to deduce that $(\Bbbk_{\gamma'} \star R\Gamma_{U_1\backslash U_0}(\mathscr{F})) = 0$. Hence $(x_0, \xi_0) \notin SS(\mathscr{F})$. □

1.4. **Microlocal Morse Theory.** As we have seen in the non-characteristic deformation lemma, the notion of singular support detects how sections of sheaves propagate/extend, if a family of open subsets $U_t\,(t \in \mathbb{R})$ does not pass $SS(\mathscr{F})$, then for $s > t$,

$$R\Gamma(U_s, \mathscr{F}) \xrightarrow{\sim} R\Gamma(U_t, \mathscr{F}).$$

This may remind us of Morse theory, where if $\varphi^{-1}([a, b])$ does not contain critical points, then one have

$$H^*(\varphi^{-1}((-\infty, b]); \Bbbk) \xrightarrow{\sim} H^*(\varphi^{-1}((-\infty, a]); \Bbbk).$$

Indeed, we can generalize Morse theory about (cohomology) of costant sheaves $\Bbbk$ to general sheaves.

Lemma 1.16. *For $\mathscr{F} \in D^b(\mathbb{R})$ such that $SS(\mathscr{F}) \cap T^*[a,b] \subset T^*_{\leq 0}[a,b]$, there is an isomorphism*

$$R\Gamma((-\infty, b], \mathscr{F}) \xrightarrow{\sim} R\Gamma((-\infty, a], \mathscr{F}).$$

This lemma follows immediately from the non-characteristic deformation lemma.

Theorem 1.17. *Let $\mathscr{F} \in D^b(M)$ and $\varphi \in C^1(M)$ a proper function such that for any $x \in \varphi^{-1}([a,b])$, $d\varphi(x) \notin SS(\mathscr{F})$. Then there is an isomorphism*

$$R\Gamma(\varphi^{-1}((-\infty, b]), \mathscr{F}) \xrightarrow{\sim} R\Gamma(\varphi^{-1}((-\infty, a]), \mathscr{F}).$$

Proof. Note that since φ is proper, we have

$$R\Gamma(\varphi^{-1}((-\infty, a]), \mathscr{F}) \simeq R\Gamma((-\infty, a], R\varphi_*\mathscr{F}).$$

The result follows from the singular support estimate $SS(R\varphi_*\mathscr{F}) \subset \varphi_\pi \varphi_d^{-1}(SS(\mathscr{F}))$ which is contained in the zero section. □

Theorem 1.18 (Morse Inequality)**.** *Let $\mathscr{F} \in D^b(M)$ and $\varphi \in C^1(M)$ be proper, and* $\mathrm{supp}(\mathscr{F}) \cap \varphi^{-1}((-\infty, t])$ *is compact. Let $\Lambda_\varphi = \{(x, d\varphi(x)) | x \in M\}$. Suppose that*

$$\Lambda_\varphi \cap SS(\mathscr{F}) = \{(x_1, \xi_1), ..., (x_n, \xi_n)\}$$

and $V_i = R\Gamma_{\varphi \geq \varphi(x_i)}(\mathscr{F})_{x_i}$ is of finite dimension. Then $R\Gamma(M, \mathscr{F})$ is of finite dimension, and

$$\sum_{j \leq l} (-1)^{l-j} \dim H^j(M, \mathscr{F}) \leq \sum_{1 \leq i \leq n} \sum_{j \leq l} \dim H^j(V_i),$$
$$\sum_{j} (-1)^{j} \dim H^j(M, \mathscr{F}) = \sum_{1 \leq i \leq n} \sum_{j} (-1)^j \dim H^j(V_i).$$

Here $x_1, ..., x_n$ are the generalization of Morse critical points, and $\dim H^j(V_i)$ is the generalization of Morse index.

Proof. Note that since φ is proper,

$$R\Gamma_{[t,+\infty)}(R\varphi_*\mathscr{F})_t = R\Gamma(\varphi^{-1}(t), R\Gamma_{\varphi^{-1}([t,+\infty))}(\mathscr{F})) = \bigoplus_{\varphi(x_i)=t} R\Gamma_{\varphi^{-1}([t,+\infty))}(\mathscr{F}))_{x_i}.$$

Therefore, again it suffices to work on $\mathbb{R}$. We consider the exact triangle

$$R\Gamma_{[t_i,+\infty)}(\mathscr{F})_{t_i} \to R\Gamma((-\infty, t_i], \mathscr{F}) \to R\Gamma((-\infty, t_i), \mathscr{F}) \xrightarrow{[1]}.$$

By non-characteristic deformation lemma we have

$$R\Gamma((-\infty, t_i], \mathscr{F}) \simeq R\Gamma((-\infty, t_{i+1}), \mathscr{F}).$$

Since $\mathrm{supp}(\mathscr{F}) \cap \varphi^{-1}((-\infty, t])$ is compact, we can prove by induction that $R\Gamma(\mathbb{R}, \mathscr{F})$ has finite dimension. Now we use the exact triangle. This will give us the inequalities we want. □

2. Constructibility

We will prove in Theorem 3.16 that the singular support of a sheaf is a coisotropic subset in T^*M. The simplest coisotropic subsets are Lagrangian subsets. We will study these sheaves in this section. In fact, such sheaves can be viewed as stratified local systems, whose singular support is contained in the conormal bundle of the stratification.

2.1. **Cohomological Constructibility.** Before talking about singular supports and stratifications, let's first introduce the following notion of a cohomologically constructible sheaf. Basically most abstract properties we need will follow from the cohomologically constructible condition.

Definition 2.1. *Let $\mathcal{C}$ be a category, and $\mathcal{I}$ be a filtrant category. Let $F_i\,(i \in \mathcal{I})$ be an inductive system in $\mathcal{C}$. Then $\varinjlim_{i\in\mathcal{I}} F_i$ is the functor*

$$\mathcal{C}^{op} \to \mathrm{Set},\ \ X \mapsto \varinjlim_{i\in\mathcal{I}} Hom_{\mathcal{C}}(X, F_i).$$

Definition 2.2. *A sheaf $\mathscr{F} \in D^b(M)$ is cohomologically constructible if*
(1). $\varinjlim_{U:x\in U} R\Gamma(U,\mathscr{F})$ and $\varprojlim_{U:x\in U} R\Gamma_c(U,\mathscr{F})$ are representable;
(2). $\varinjlim_{U:x\in U} R\Gamma(U,\mathscr{F}) \xrightarrow{\sim} \mathscr{F}_x$ and $R\Gamma_{\{x\}}(\mathscr{F}) \xrightarrow{\sim} \varprojlim_{U:x\in U} R\Gamma_c(U,\mathscr{F})$;
(3). $\mathscr{F}_x$ and $R\Gamma_{\{x\}}(\mathscr{F})$ are perfect.

Proposition 2.1. *Let $\mathscr{F} \in D^b(M)$ be cohomologically constructible. Then*
(1). $D\mathscr{F}$ is cohomologically constructible;
(2). $\mathscr{F} \mapsto DD\mathscr{F}$ is an isomorphism;
(3). $R\Gamma_{\{x\}}(M, D\mathscr{F}) = RHom(\mathscr{F}_x, \Bbbk)$ and $(D\mathscr{F})_x = RHom(R\Gamma_{\{x\}}(M,\mathscr{F}), \Bbbk)$.

Proposition 2.2. *Let $\mathscr{F} \in D^b(M)$ be cohomologically constructible. Then*

$$D\mathscr{F} \boxtimes \mathscr{G} \simeq R\mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^{!}\mathscr{G}),$$
$$D'\mathscr{F} \boxtimes \mathscr{G} \simeq R\mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^{-1}\mathscr{G}).$$

Proposition 2.3. *Let $\mathscr{F}, \mathscr{G} \in D^b(M)$ be cohomologically constructible. Then*

$$R\mathscr{H}om(\mathscr{F},\mathscr{G}) \simeq R\mathscr{H}om(D\mathscr{F}, D\mathscr{G}) \simeq D\left(D\mathscr{F} \otimes \mathscr{G}\right).$$

2.2. **Subanalytic Stratification.** Now we define what a constructible sheaf is. Basically it is a stratified local system. In order to state the definition, we should first explain what kind of stratification we will be considering - it is the subanalytic stratification.

Definition 2.3. *A subset $Z \subset M$ is subanalytic at $x \in M$ if there exists an open neighbourhood U of x, and compact manifolds $Y_j^i\ (i = 1, 2, 1 \le j \le N)$ and $f_j^i : Y_j^i \to M$ analytic functions such that*

$$Z \cap U = U \cap \left(\bigcup_{1\le j\le N} f_j^1(Y_j^1) \backslash f_j^2(Y_j^2) \right).$$

Z is a subanalytic set if it is subanalytic at any point.

Lemma 2.4 (Curve selection lemma). *Let $Z \subset M$ be subanalytic and $x_0 \in \overline{Z}$. Then there exists an analytic path $x : [0, 1] \mapsto M$ so that $x(0) = x_0$ and $x((0, 1]) \subset Z$.*

Definition 2.4. *(1). A partition $M = \bigsqcup_{i\in I} M_i$ is called a subanalytic stratification if it is locally finite, all M_i's are subanalytic subsets and for any $i, j \in I$, $\overline{M}_i \cap M_j \neq \emptyset$ iff $M_j \subset \overline{M}_i$.*
(2). A partition $M = \bigsqcup_{i\in I} M_i$ is called a μ-stratification if it is subanalytic and for any $i, j \in I$, $M_j \subset \overline{M}_i \backslash M_i$, we have

$$\left(T^*_{M_i}M \hat{+} T^*_{M_j}M\right) \cap \pi^{-1}(M_j) \subset T^*_{M_j}M.$$

Theorem 2.5. *Let $M = \bigcup_{i\in I} M_i$ be a locally finite subanalytic covering. Then there exists a refinement $M = \bigsqcup_{i'\in I'} M'_{i'}$ being a μ-stratification.*

Definition 2.5. *Let $\mathscr{F} \in D^b(M)$. $\mathscr{F}$ is weakly $\mathbb{R}$-constructible if there exists a locally finite covering $M = \bigcup_{i\in I} M_i$ by subanalytic sets such that for any $i \in I, j \in \mathbb{Z}$, $H^j\mathscr{F}|_{M_i}$ is locally constant. The full subcategory of weakly $\mathbb{R}$-constructible sheaves is $D^b_{\mathbb{R}\text{-}con,weak}(M)$.*

$\mathscr{F}$ is $\mathbb{R}$-constructible if in addition for any $x \in M$, $\mathscr{F}_x$ is perfect. The full subcategory of $\mathbb{R}$-constructible sheaves is $D^b_{\mathbb{R}\text{-}con}(M)$.

The following theorem characterises the microlocal behaviour of a constructible sheaf. In short, a constructible sheaf is a sheaf with singular support being a conical Lagrangian.

Theorem 2.6. *Let $\mathscr{F} \in D^b(M)$. Then the following are equivalent:*
(1). $\mathscr{F} \in D^b_{\mathbb{R}\text{-}con,weak}(M)$;
(2). $SS(\mathscr{F})$ is contained in a closed subanalytic isotropic subset;
(3). $SS(\mathscr{F})$ is a closed conical Lagrangian subset.

Proposition 2.7. *Let $\mathscr{F} \in D^b(M)$ and $M = \bigsqcup_{i\in I} M_i$ be a stratification by subanalytic sets. Then the following are equivalent:*
(1). for all $i \in I, j \in \mathbb{Z}$, $H^j\mathscr{F}|_{M_i}$ is locally constant;
(2). $SS(\mathscr{F}) \subset \bigsqcup_{i\in I} T^*_{M_i}M$.

Now our goal is to show that the notion of $\mathbb{R}$-constructibility implies cohomologically constructibility we introduced in the previous section. First we need a simple lemma which is essentially Sard theorem.

Lemma 2.8 (Microlocal Bertini-Sard theorem)**.** *Let $\varphi \in C^1(M)$ be proper $\Lambda \subset T^*M$ be a closed conical subanalytic isotropic set. Then $S = \{t \in \mathbb{R} | \exists\, x \in M, \varphi(x) = t, d\varphi(x) \in \Lambda\}$ is discrete.*

Proposition 2.9. *Let $\mathscr{F} \in D^b_{\mathbb{R}\text{-}con}(M)$. Then $\mathscr{F}$ is cohomologically constructible.*

Proof. First we show representability and the isomorphisms

$$\varinjlim_{U: x\in U} R\Gamma(U, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}_x, \;\; R\Gamma_{\{x\}}(\mathscr{F}) \xrightarrow{\sim} \varprojlim_{U: x\in U} R\Gamma_c(U, \mathscr{F}).$$

Choose a proper real analytic function $\varphi : M \to \mathbb{R}^n$. We show that there is a natural isomorphism for $\epsilon > 0$ small enough,

$$R\Gamma(\varphi^{-1}(\overline{B_\epsilon(0)}), \mathscr{F}) \xrightarrow{\sim} R\Gamma(\varphi^{-1}(0), \mathscr{F}),$$
$$R\Gamma_{\varphi^{-1}(0)}(\mathscr{F}) \xrightarrow{\sim} R\Gamma_c(\varphi^{-1}(0), \mathscr{F}).$$

Let $\Lambda = SS(\mathscr{F})$ and apply the microlocal Bertini-Sard theorem, then these isomorphisms hold by microlocal cut-off lemma.

Then it suffices to show that $R\Gamma_{\{x\}}(M, \mathscr{F})$ is perfect. In fact consider the exact triangle

$$R\Gamma_{\{x\}}(M, \mathscr{F}) \to R\Gamma(B_\epsilon(x), \mathscr{F}) \to R\Gamma(B_\epsilon(x)\backslash\{x\}, \mathscr{F}) \xrightarrow{[1]} .$$

One can prove $R\Gamma(B_\epsilon(x)\backslash\{x\}, \mathscr{F}) \simeq R\Gamma(S_{\epsilon/2}(x), \mathscr{F})$, but since the projection by the radius function is proper, constructibility is preserved and hence $R\Gamma(S_{\epsilon/2}(x), \mathscr{F})$ is perfect. On the other hand, $R\Gamma(B_\epsilon(x), \mathscr{F}) \simeq \mathscr{F}_x$ is also perfect. Thus we are done. □

One can in fact find the generators of the category of constructible sheaves.

Theorem 2.10 (Nadler)**.** *Let $\mathcal{T} = \{\tau_\alpha | \alpha \in I\}$ be a subanalytic triangulation, $D^b_{\mathcal{T}}(M)$ be the derived category of $\mathcal{T}$-constructible sheaves. Let $\mathcal{C}_{\mathcal{T}}(M)$ be the full subcategory of $j_{\alpha,*}\Bbbk_{\tau_\alpha}$. Then $D^b_{\mathcal{T}}(M)$ is the triangulated envelope of $\mathcal{C}_{\mathcal{T}}(M)$.*

Proof. Let $i_{\geq k}: T_{\geq k} \to M$ be the inclusion of all simplices with dimension greater than or equal to k, and $j_{<k}: T_{<k} \to M$ be the inclusion of all simplices with dimension less than k. Let $D^b_{\mathcal{T}_{\geq k}}(M)$ be the subcategory of sheaves $\mathscr{F} \simeq i_{\geq k,*}i^{-1}_{\geq k}\mathscr{F}$.

It is easy to realize that $D^b_{\mathcal{T}_{\geq n}}(M)$ is generated by $j_{\alpha,*}\Bbbk_{\tau_\alpha}$. Suppose we already know for $l > k$ $D^b_{\mathcal{T}_{\geq l}}(M)$ is generated by $j_{\alpha,*}\Bbbk_{\tau_\alpha}$. Then we consider $D^b_{\mathcal{T}_{\geq k}}(M)$. Note that we have the following exact triangle

$$R\Gamma_{T_{<k+1}}(\mathscr{F}) \to \mathscr{F} \to i_{\geq k+1,*}i^{-1}_{\geq k+1}\mathscr{F} \xrightarrow{[1]} .$$

Since $\mathscr{F} \simeq i_{\geq k,*}i^{-1}_{\geq k}\mathscr{F}$ we know by the exact triangle

$$R\Gamma_{T_{<k}}(\mathscr{F}) \to \mathscr{F} \to i_{\geq k,*}i^{-1}_{\geq k}\mathscr{F} \xrightarrow{[1]}$$

that $R\Gamma_{T_{<k}}(\mathscr{F}) \simeq 0$. Now apply $R\Gamma_{T_{<k}}$ to the first exact triangle we know that

$$R\Gamma_{T_{<k}}R\Gamma_{T_{<k+1}}(\mathscr{F}) \simeq 0.$$

However by the second exact triangle this means that $R\Gamma_{T_{<k+1}}(\mathscr{F})$ is in $D^b_{\mathcal{T}_{\geq k}}(M)$ and hence must be supported on dimension k simplices. Therefore by induction $D^b_{\mathcal{T}}(M)$ is generated by $j_{\alpha,*}\Bbbk_{\tau_\alpha}$. □

Similarly if one consider $\mathcal{C}'_{\mathcal{T}}(M)$ consisting of $j_{\alpha,!}\Bbbk_{\tau_\alpha}$ the same result holds.

3. Microlocalization

Denote by $D^b_\Lambda(M)$ the full subcategory of $D^b(M)$ whose singular support is in $\Lambda \subset T^*M$. Consider $\Lambda \subset T^*M$. The motivation of microlocalization is to focus on the behaviour of sheaves along Λ. Therefore we enforce all the sheaves in $D^b_{M\cup T^*M\backslash\Lambda}(M)$ to be zero, which means we consider the localization category

$$D^b(M,\Lambda) = D^b(M)/D^b_{M\cup T^*M\backslash\Lambda}(M).$$

and $\mu hom(\mathscr{F},\mathscr{G}) = Hom_{D^b(M,\Lambda)}(\mathscr{F},\mathscr{G})$.

Alternatively, one can avoid using localization of categories and explicitly define microlocalization and the functor μhom.

3.1. **Specialization.** Let $N \subset M$ be a submanifold. Consider the normal deformation $\tilde{M}_N$ of M along N, so that there is a projection $p: \tilde{M}_N \to M$. $p^{-1}(M\backslash N) = (M\backslash N) \times (\mathbb{R}\backslash\{0\})$, $p^{-1}(N) = T_NM \cup N \times \mathbb{R}$. There is also a projection $t: \tilde{M}_N \to \mathbb{R}$ so that $t|_{p^{-1}(N)} = 0$, $t|_{p^{-1}(M\backslash N)}$ is the canonical projection.

Locally, let $\{U_i\}_{i\in I}$ be an open cover of M, $\varphi_i: U_i \to \mathbb{R}^n$ be coordinate systems so that $U_i \cap N = \varphi^{-1}(0 \times \mathbb{R}^{n-k})$. Define

$$V_i = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} | (tx', x'') \in \varphi_i(U_i)\},$$

and the transition functions g_{ij} satisfy

$$(tg'_{ij}(x,t), g''_{ij}(x,t)) = \varphi_j \circ \varphi_i^{-1}(tx', x'').$$

The normal bundle T_NM is blown up to encode the information of normal directions to N. t is a rescaling factor to shrink the distance from a point to N, so that one doesn't need to worry about the influence on the normal direction coming from the distance factor.

Furthermore, we should make a remark here that in fact the normal deformation is given by the real blowup, $\tilde{M}_N = Bl_{N\times 0}(M \times \mathbb{R})\backslash Bl_NM$.

Write $\tilde{M}_{N+} = t^{-1}((0,+\infty))$, $p_+: \tilde{M}_{N,+}$ the restriction of $p: \tilde{M} \to M$. There is a diagram

$$T_NM \xrightarrow{s} \tilde{M}_N \xleftarrow{j} \tilde{M}_{N,+}.$$

FIGURE 1. When $M=\mathbb{R}^2$, $N=\{0\}$, $V=\{(r,\theta)|\theta<\pi/10\}$ on the left and open subsets U such that $C_N(M\backslash U)\cap V=\emptyset$ on the right.

Definition 3.1. *Let $S\subset M$ be locally closed. Then the Whitney normal cone along N is $C_N(S)=\overline{p_+^{-1}(S)}\cap T_NM$. Let $S_1,S_2\subset M$ be locally closed. Then the Whitney normal cone is $C(S_1,S_2)=C_\Delta(S_1\times S_2)$.*

Basically, $C_N(S)$ is the limit points of $p_+^{-1}(S)$ in T_NM. When a point $(x,t)\in p_+^{-1}(S)$ approaches the normal bundle, this means under the projection $(tx',x'')\in M$ approaches N along the corresponding normal direction. One can see that the factor t shrinks the distance between the point and N. The following proposition is elementary.

Proposition 3.1. *Let $x=(x',x'')$ be a local coordinate system on M such that N is defined by $x'=0$. Then $(x,\xi)\in C_N(S)$ iff there exists a sequence (x_n',x_n'',c_n) in $S\times\mathbb{R}_+$ such that*

$$(x_n',x_n'')\to x,\ \ c_nx_n'\to\xi.$$

Now we define what the specialization of a sheaf is. The idea is that, we want to detect the behaviour of a sheaf after contracting everything to an infinitesimal normal neighbourhood of N. Therefore we pull the sheaf back to the normal deformation along N and focus on the behaviour on the normal bundle T_NM.

Definition 3.2. *Let $N\subset M$, $\mathscr{F}\in D^b(M)$. Then the specialization of $\mathscr{F}$ along N is*

$$\nu_N\mathscr{F}=s^{-1}Rj_*p_+^{-1}\mathscr{F}.$$

Theorem 3.2. *Let $V\subset T_NM$ be an open conical subset. Then*

$$H^j(V,\nu_N\mathscr{F})=\varinjlim_{U:C_N(M\backslash U)\cap V=\emptyset}H^j(U,\mathscr{F}).$$

Let's first look at an example (Figure 1) to see what this theorem says. Consider $M=\mathbb{R}^2$, $N=\{0\}$, $V=\{(r,\theta)|\theta<\pi/10\}$. Then $C_N(M\backslash U)\cap V=\emptyset$ means the points in $M\backslash U$ approach V along a direction away from V, so $M\backslash U$ is away from V infinitesimally. In this case, the theorem says the sections of $\nu_N\mathscr{F}$ on V are direct limit of sections of $\mathscr{F}$ on certain neighbourhoods of V.

Proof of Theorem 3.2. Let $U\subset M$ be an open subset such that $C_N(M\backslash U)\cap V=\emptyset$. Note that $p_+^{-1}(U)\cup V$ is a neighbourhood of V (see Figure 2). There are natural morphisms

$$\begin{aligned}R\Gamma(U,\mathscr{F})&\to R\Gamma(p^{-1}(U),p^{-1}\mathscr{F})\to R\Gamma(p^{-1}(U)\cap\tilde{M}_{N,+},p^{-1}\mathscr{F})\\&\to R\Gamma(p_+^{-1}(U)\cup V,Rj_*j^{-1}p^{-1}\mathscr{F})\to R\Gamma(V,\nu_N\mathscr{F}).\end{aligned}$$

Therefore one gets a canonical morphism

$$\varinjlim_{U:C_N(M\backslash U)\cap V=\emptyset}H^j(U,\mathscr{F})\to H^j(V,\nu_N\mathscr{F}).$$

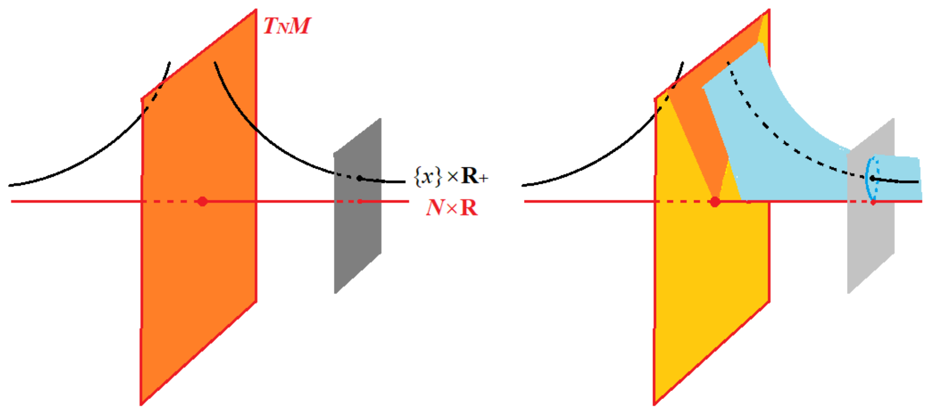

FIGURE 2. Let $M = \mathbb{R}^2, N = \{0\}$. The left hand side is $\tilde{M}_N$, where the red region is $p^{-1}(N)$, the grey slice is a copy of M, and the black curves are two components of $p^{-1}(x) \simeq \{x\} \times \mathbb{R}^\times$. Let $U \subset M, V \subset T_N M$ be as in the previous figure so that $C(M\backslash U) \cap V = \emptyset$. The right hand side shows $p_+^{-1}(U)$ as the blue region and a cone $V \subset T_N M$ as the orange region. One can see why $p_+^{-1}(U) \cup V$ is a neighbourhood of V.

On the other hand, we know that

$$H^j(V, \nu_N \mathscr{F}) = \varinjlim_{W \supset V} H^j(W, Rj_* j^{-1} p^{-1} \mathscr{F}) = \varinjlim_{W \supset V} H^j(W \cap \tilde{M}_{N,+}, p^{-1} \mathscr{F}).$$

If the neighbourhoods W so that $p|_{W \cap \tilde{M}_{N,+}}$ has connected fibers form a basis of V, then we may assume that when taking the direct limit, $p|_{W \cap \tilde{M}_{N,+}}$ always has contractible fibers. By the non-characteristic deformation lemma, $Rp_* p^{-1} \mathscr{F} = \mathscr{F}$. Consequently

$$H^j(W \cap \tilde{M}_{N,+}, p^{-1}\mathscr{F}) = H^j(p(W \cap \tilde{M}_{N,+}), Rp_* p^{-1} \mathscr{F}) = H^j(p(W \cap \tilde{M}_{N,+}), \mathscr{F}).$$

As all the subsets U such that $C_N(M\backslash U) \cap V = \emptyset$ are of the form $p(W \cap \tilde{M}_{N,+})$, we're done.

It suffices to check that the neighbourhoods W so that $p|_{W \cap \tilde{M}_{N,+}}$ has contractible fibers form a basis of V. Pick any neighbourhood W_0 of V. We consider $S\tilde{M}_N = (\tilde{M}_N \backslash (M \times \mathbb{R}))/\mathbb{R}_+$. By choosing a section on $S\tilde{M}_N$, we can choose a connected component for each fiber of $W_0 \cap (\tilde{M}_N \backslash (M \times \mathbb{R}))$. Let the union be W_1. W_1 is an open neighbourhood of $V \cap (T_N M \backslash M)$. To get an open neighbourhood of V, we just let $W = V \cup W_1 \cup (W_0 \cap t^{-1}((-\infty, 0)))$. This completes the proof. $\square$

Proposition 3.3. *Let $\mathscr{F} \in D^b(M)$. Then there is an exact triangle*

$$R\Gamma_N(\mathscr{F})|_N \to \mathscr{F}|_N \to R\dot{\pi}_* \nu_N \mathscr{F} \xrightarrow{[1]} .$$

Proof. Consider the exact triangle

$$R\Gamma_N(\nu_N \mathscr{F})|_N \to R\pi_* \nu_N \mathscr{F} \to R\dot{\pi}_* \nu_N \mathscr{F} \xrightarrow{[1]} .$$

It suffices to show that there are canonical isomorphisms

$$R\Gamma_N(\nu_N \mathscr{F})|_N \simeq R\Gamma_N(\mathscr{F})|_N, \; R\pi_* \nu_N \mathscr{F} \simeq \mathscr{F}|_N.$$

FIGURE 3. On the left is a constant sheaf supported on the interior of the standard cusp (the grey region, boundary not included), and on the right is its specialization along the origin (the red point).

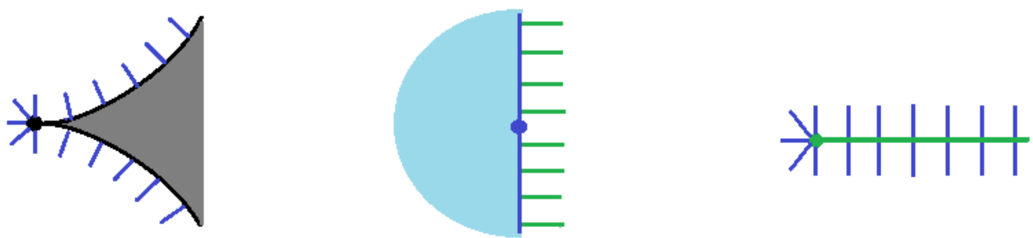

FIGURE 4. 1. On the left is the singular support of a constant sheaf supported on the interior of the standard cusp (the grey region, boundary not included); 2. in the middle is the normal cone of its singular support, where the blue color is representing the components in the base and the green color is representing components in the fibers; 3. on the right is the singular support of its specialization along the origin, where now green stands for components in the base and blue stands for components in the fibers. The blue components and the green components are corresponding to each other by dualization.

Let $i : N \to T_N M$ be the zero section. Then by the definition of specialization and the previous theorem we have

$$\mathscr{F}_N \simeq i^{-1}s^{-1}\pi^{-1}\mathscr{F} \xrightarrow{\sim} i^{-1}s^{-1}Rj_*j^{-1}\pi^{-1}\mathscr{F} \simeq \nu_N\mathscr{F}|_N.$$

Since $\pi : T_N M \to N$ has contractible fiber, we know

$$\nu_N\mathscr{F}|_N \simeq i^{-1}\nu_N\mathscr{F} \xrightarrow{\sim} R\pi_*\pi^{-1}i^{-1}\nu_N\mathscr{F} \simeq R\pi_*\nu_N\mathscr{F}.$$

This proves the first isomorphism. The second isomorphism is similar as $R\Gamma_N(\mathscr{F})|_N \simeq i^{-1}\mathscr{F}$. □

In addition, we can estimate the singular support of the specialization in terms of the singular support of the original sheaf. An example is illustrated in Figure 3 and 4.

Proposition 3.4. *Let $\mathscr{F} \in D^b(M)$. Then*

$$SS(\nu_N\mathscr{F}) \subset C_{T^*_N M}(SS(\mathscr{F})).$$

Proof. Fix a local chart U so that on that local chart $M = \mathbb{R}^n, N = 0 \times \mathbb{R}^{n-k}$. Locally the normal deformation is $\tilde{M}_N = \mathbb{R}^n \times \mathbb{R}$, where $t : \tilde{M}_N \to \mathbb{R}$; $(x', x'', t) \mapsto t$, and $p : \tilde{M}_N \to$

FIGURE 5. $SS(\Bbbk_{\mathbb{R}}), SS(\Bbbk_0), SS(\Bbbk_{(0,\infty)})$ and $SS(\Bbbk_{[0,\infty)})$. The grey line is the zero section $\mathbb{R} \subset T^*\mathbb{R}$.

$\mathbb{R}$; $(x', x'', t) \mapsto (tx', x'')$. We have

$$SS(p_+^{-1}\mathscr{F}) = \{(x, t; \xi, \tau) | t > 0, t\tau - \langle x', \xi' \rangle = 0, (tx', x''; t^{-1}\xi', \xi'') \in SS(\mathscr{F})\}.$$

For any $(x_0', x_0''; \xi_0', \xi_0'') \in SS(\nu_M \mathscr{F})$, there exists $(x_n', x_n'', t_n; \xi_n', \xi_n'', \tau_n) \in SS(p_+^{-1}\mathscr{F})$ so that

$$t_n \to 0,\ t_n\tau_n \to 0,\ (x_n', x_n''; \xi_n', \xi_n'') \to (x_0', x_0''; \xi_0', \xi_0'').$$

Since $(t_n x_n', x_n''; t_n^{-1}\xi_n', \xi_n'') \in SS(\mathscr{F})$, we know (by scaling) $(t_n x_n', x_n''; \xi_n', t_n\xi_n'') \in SS(\mathscr{F})$. Therefore $(x_0', x_0''; \xi_0', \xi_0'') \in C_{T_N^*M}(SS(\mathscr{F}))$. □

3.2. **Microlocalization.** For microlocalization, instead of working with normal bundles, we work with conormal bundles. This means we will nee to pass from a vector bundle to its dual vector bundle. The corresponding transformation on sheaves is call Fourier-Sato transform.

Definition 3.3. *Let $\pi : E \to M$ be a vector bundle and $\pi^\vee : E^\vee \to M$ its dual bundle, $p_1 : E \oplus E^\vee \to E, p_2 : E \oplus E^\vee \to E^\vee$. Let*

$$D = \{(x, u, v^\vee) | \langle u, v^\vee \rangle \le 0\}.$$

Then the Fourier-Sato transform of $\mathscr{F} \in D^b(E)$ is

$$\mathscr{F}^\wedge = Rp_{2,!}(p_1^{-1}\mathscr{F})_D.$$

Let $D' = \{(x, u, v^\vee) | \langle u, v^\vee \rangle \ge 0\}$. Then the inverse Fourier-Sato transform of $\mathscr{G} \in D^b(E^\vee)$ is $\mathscr{G}^\vee = Rp_{1,!}(p_2^!\mathscr{G})_{D'}$.

Let's consider some examples. Let $M = \mathrm{pt}$ and $E = E^\vee = \mathbb{R}$. Then $D = \{(x, y) \in \mathbb{R}^2 | xy \le 0\}$. Let's compute $\Bbbk^\vee = Rp_{2,!}(p_1^{-1}\Bbbk)_D = Rp_{2,!}\Bbbk_D$. For $y \ne 0$, its stalk is

$$(\Bbbk^\wedge)_y = R\Gamma_c(p_2^{-1}(y), \Bbbk_D) = H_c^*([0, +\infty); \Bbbk) = 0.$$

For $y = 0$, its stalk is

$$(\Bbbk^\wedge)_0 = R\Gamma_c(p_2^{-1}(0), \Bbbk_D) = H_c^*(\mathbb{R}; \Bbbk) = \Bbbk[-1].$$

Therefore, $\Bbbk^\wedge = \Bbbk_0[-1]$.

For the skyscraper sheaf $\Bbbk_0$, let's also compute $\Bbbk_0^\vee = Rp_{2,!}(p_1^{-1}\Bbbk_0)_D = Rp_{2,!}\Bbbk_{\{(x,y)|x=0\}}$. For $y \in \mathbb{R}$, its stalk is

$$(\Bbbk^\wedge)_y = R\Gamma_c(p_2^{-1}(y), \Bbbk_{\{(x,y)|x=0\}}) = H_c^*(\mathrm{pt}; \Bbbk) = \Bbbk.$$

Therefore, $\Bbbk_0^\wedge = \Bbbk$.

Similarly, we can also get $\Bbbk_{[0,\infty)}^\wedge = \Bbbk_{(0,\infty)}$, and $\Bbbk_{(0,\infty)}^\wedge = \Bbbk_{(-\infty,0]}[-1]$. Therefore, when considering the singular support of these sheaves, in dimension 1, we can see that Fourier-Sato transform is actually rotating the singular support by 90 degrees.

Before stating any propositions, let's recall the following notation. Let $\gamma \subset E$ be a cone. Then

$$\gamma^\vee = \{v \in E^\vee | \, \langle u, v\rangle \geq 0, \, \forall \, u \in \gamma\}.$$

Lemma 3.5. *Let γ be a closed proper convex cone containing the zero section. Then*

$$(\Bbbk_\gamma)^\wedge \simeq \Bbbk_{(\gamma^\vee)^\circ}.$$

Let γ be an open convex cone. Then

$$(\Bbbk_\gamma)^\wedge \simeq \Bbbk_{-\gamma^\vee} \otimes or_{E^\vee}[-n].$$

Proof. We only prove the first isomorphism. In fact, for any $y \in E^\vee$,

$$\big((\Bbbk_\gamma)^\wedge\big)_y = R\Gamma_c(p_2^{-1}(y), \Bbbk_{(\gamma \times E^\vee) \cap D}) = R\Gamma_c(p_1(p_2^{-1}(y) \cap D) \cap \gamma, \Bbbk).$$

When $y \notin (\gamma^\vee)^\circ$, $p_1(p_2^{-1}(y) \cap D) \cap \gamma$ is a closed proper cone, so

$$R\Gamma_c(p_1(p_2^{-1}(y) \cap D) \cap \gamma, \Bbbk) \simeq H_c^*(\mathbb{R}^n_{\geq 0}; \Bbbk) = 0.$$

When $y \in (\gamma^\vee 0^\circ$, $p_1(p_2^{-1}(y) \cap D) \cap \gamma = 0$. Hence

$$R\Gamma_c(p_1(p_2^{-1}(y) \cap D) \cap \gamma, \Bbbk) \simeq H_c^*(\mathrm{pt}; \Bbbk) = \Bbbk.$$

This shows $(\Bbbk_\gamma)^\wedge \simeq \Bbbk_{(\gamma^\vee)^\circ}$. □

Theorem 3.6. *The Fourier-Sato transform induces an equivalence $D^b(E) \to D^b(E^\vee)$. In particular,*

$$RHom(\mathscr{F}, \mathscr{G}) = RHom(\mathscr{F}^\wedge, \mathscr{G}^\wedge).$$

Proof. It suffices to show that the Fourier-Sato transformation gives an equivalence of categories. In fact we show that

$$\mathscr{F} \to \mathscr{F}^{\wedge\vee}$$

is an equivalence. For any conical open subset $U \subset E$, by adjunction we have

$$H^j(U, \mathscr{F}^{\wedge\vee}) = Hom(\Bbbk_U, \mathscr{F}^{\wedge\vee}[j]) = Hom((\Bbbk_U)^{\wedge\vee}, \mathscr{F}[j]).$$

Therefore, it suffices to check that $\Bbbk_U \to (\Bbbk_U)^{\wedge\vee}[n]$ is an equivalence. □

Proposition 3.7. *Let $\gamma \subset E^\vee$ be a closed proper cone containing the zero section. Then*

$$R\Gamma_\gamma(E^\vee, \mathscr{F}^\wedge) \simeq R\Gamma((-\gamma^\vee)^\circ, \mathscr{F}) \otimes \omega_{E/M},$$

where $\gamma^\vee = \{x \in E | \, \langle x, y\rangle \geq 0, y \in \gamma\}$.

Proof. The result follows from the fact that $R\Gamma_\gamma(E^\vee, \mathscr{F}) = RHom(\Bbbk_\gamma, \mathscr{F})$. □

We estimate the singular support of Fourier-Sato transform in the following proposition.

Proposition 3.8. *Let E be a vector space (vector bundle), $\mathscr{F} \in D^b(M)$. Then*

$$SS(\mathscr{F}^\wedge) = SS(\mathscr{F}).$$

Proof. Since Fourier-Sato transformation is an equivalence, it suffices to show that $SS(\mathscr{F}^\wedge) \subset SS(\mathscr{F})$. Suppose $(x, \xi) \notin SS(\mathscr{F})$. We will show that $(\xi, -x) \notin SS(\mathscr{F}^\wedge)$.

Without loss of generality, we assume that E is a vector space. First suppose

$$R\Gamma_0(\mathscr{F}) = 0.$$

Denote $j : E\backslash 0 \to E$, $\tilde{j} : (E\backslash 0) \times E^\vee \to E \times E^\vee$ and $\tilde{p}_2 : (E\backslash 0) \times E^\vee \to E^\vee$. Then $Rj_*j^{-1}\mathscr{F} \simeq \mathscr{F}$.

$$\begin{aligned}\mathscr{F}^\wedge &\simeq Rp_{2,!}R\Gamma_{D'}(p_1^{-1}Rj_*j^{-1}\mathscr{F}) = Rp_{2,!}R\Gamma_{D'}(R\tilde{j}_*\tilde{j}^{-1}p_1^{-1}\mathscr{F}) \\ &= Rp_{2,!}R\tilde{j}_!\tilde{j}^{-1}R\Gamma_{D'}(p_1^{-1}\mathscr{F}) = R\tilde{p}_{2,!}\tilde{j}^{-1}R\Gamma_{D'}(p_1^{-1}\mathscr{F}).\end{aligned}$$

If $(\xi, -x) \in SS(\mathscr{F}^\wedge)$, then by Proposition, there exists $y \in E^\vee$ such that $(y, \xi; 0, -x) \in SS(R\Gamma_{D'}(p_1^{-1}\mathscr{F}))$. Now since $R\Gamma_{D'}(p_1^{-1}\mathscr{F}) = R\mathscr{H}om(\Bbbk_{D'}, p_1^{-1}\mathscr{F})$, by Proposition there exists $\eta \in T_y^*E^\vee$ such that

$$(y, \xi; \eta, 0) \in SS(p_1^{-1}\mathscr{F}),\ \ (y, \xi; \eta, x) \in SS(\Bbbk_{D'}).$$

Thus $(y, \eta) \in SS(\mathscr{F})$, $x = \lambda y$ and $\xi = \lambda \eta$. However, $SS(\mathscr{F})$ is invariant under the $\mathbb{R}_+$-action $(x; \xi) \mapsto (\lambda x, \lambda^{-1}\xi)$. This shows a contradiction.

In general, if $R\Gamma_0(\mathscr{F}) \neq 0$, then we consider the exact triangle

$$R\Gamma_0(\mathscr{F}) \to \mathscr{F} \to Rj_*j^{-1}\mathscr{F} \xrightarrow{[1]} .$$

It suffices to show that $(\xi, -x) \notin SS(R\Gamma_0(\mathscr{F})^\wedge)$. Write $i : 0 \to E$. We have $R\Gamma_0(\mathscr{F})^\wedge \simeq (i_!i^{-1}\mathscr{F})^\wedge$. Since $i_!i^{-1}\mathscr{F}$ is supported at 0, we know it is a skyscraper sheaf supported at 0. Therefore $(i_!i^{-1}\mathscr{F})^\wedge$ is a constant sheaf on $E^\vee$. This shows that as long as $x \neq 0$, $(\xi, -x) \notin SS(R\Gamma_0(\mathscr{F})^\wedge)$.

Finally to resolve the issue that $x = 0$ or $\xi = 0$, we add an extra factor $\Bbbk_{\mathbb{R}\times 0} \in D^b(\mathbb{R}^2)$. Consider

$$\mathscr{F} \boxtimes \Bbbk_{\mathbb{R}\times 0} = \pi_E^{-1}\mathscr{F} \otimes \pi_{\mathbb{R}^2}^{-1}\Bbbk_{\mathbb{R}\times 0}.$$

Then apply the previous argument. Since $SS(\Bbbk_{\mathbb{R}\times 0}) = \{(x, 0; 0, \eta) | x, \eta \in \mathbb{R}\}$, we are through. □

Now we are able to define microlocalization in terms of the Fourier-Sato transform of the specialization.

Definition 3.4. *Let $N \subset M$, $\mathscr{F} \in D^b(M)$. The microlocalization along N is $\mu_N\mathscr{F} = \nu_N\mathscr{F}^\wedge \in D^b(T_N^*M)$.*

The following theorems follow from the properties of specialization and Fourier-Sato transform.

Theorem 3.9. *Let $V \subset T_NM$ be an open conical subset, $\mathscr{F} \in D^b(M)$. Then*

$$H^j(V, \mu_N\mathscr{F}) = \varinjlim_{U: U\cap N = \pi(V);\, Z: C_N(Z) \subset V^\vee} H^j_{U\cap Z}(U, \mathscr{F}).$$

Proposition 3.10. *Let $\mathscr{F} \in D^b(M)$. Then there is an exact triangle*

$$\mathscr{F}|_N \otimes \omega_{N/M} \to R\Gamma_N(\mathscr{F})|_N \to R\dot{\pi}_*\mu_N\mathscr{F} \xrightarrow{[1]} .$$

Proof. Consider the following exact triangle

$$R\Gamma_N(\mu_N\mathscr{F}) \to R\pi_*\mu_N\mathscr{F} \to R\dot{\pi}_*\mu_N\mathscr{F} \xrightarrow{[1]} .$$

We prove that there are natural isomorphisms $R\Gamma_N(\mu_N\mathscr{F}) \simeq \mathscr{F}|_N \otimes \omega_{N/M}$, $R\pi_*\mu_N\mathscr{F} \simeq R\Gamma_N(\mathscr{F})|_N$. □

Proposition 3.11. *Let $\mathscr{F} \in D^b(M)$. Then*

$$SS(\mu_N\mathscr{F}) \subset C_N(SS(\mathscr{F})).$$

3.3. **The Functor *μhom*.** In this subsection we define the functor *μhom* by microlocalization along the diagonal.

Definition 3.5. *Let $\delta : T^*M \to T^*_\Delta(M \times M)$ be the isomorphism $(x, \xi) \mapsto (x, x, \xi, -\xi)$. Then for $\mathscr{F}, \mathscr{G} \in D^b(M)$,*

$$\mu hom(\mathscr{F}, \mathscr{G}) = \delta^{-1}\mu_\Delta R\mathscr{H}om(p_1^{-1}\mathscr{F}, p_2^!\mathscr{G}).$$

Proposition 3.12. *Let E be a vector space (vector bundle), $\mathscr{F},\mathscr{G}\in D^b(E)$. Let $(x,\xi)\in T^*E$. Then*

$$H^j\mu hom(\mathscr{F},\mathscr{G})_{(x,\xi)}=\varinjlim_{U:x\in U;\,\gamma\subset(\{\xi\}^\vee)^\circ\cup\{0\}} H^j(R\Gamma(U,R\mathscr{H}om(\Bbbk_\gamma\star\mathscr{F}_U,\mathscr{G}))).$$

Proof. Let $\gamma\subset E$ be a proper closed cone. We can define

$$Z_\gamma=\{(x,x')\in E\times E|x-x'\in\gamma\}.$$

Then $C_\Delta(Z_\gamma)\subset\{\xi\}^\vee$. Hence as the open subsets $U\times U$ give a neighbourhood system of (x,x), by theorem 3.9,

$$\begin{aligned}H^j\mu hom(\mathscr{F},\mathscr{G})_{(x,\xi)}&=\varinjlim_{U:x\in U;\,\gamma\subset(\{\xi\}^\vee)^\circ\cup\{0\}} H^j(R\Gamma_{Z_\gamma}(U\times U,R\mathscr{H}om(p_2^{-1}\mathscr{F},p_1^!\mathscr{G})))\\&=\varinjlim_{U:x\in U;\,\gamma\subset(\{\xi\}^\vee)^\circ\cup\{0\}} H^j(R\Gamma(U\times X,R\mathscr{H}om((p_2^{-1}\mathscr{F}_U)_{Z_\gamma},p_1^!\mathscr{G})))\\&=\varinjlim_{U:x\in U;\,\gamma\subset(\{\xi\}^\vee)^\circ\cup\{0\}} H^j(R\Gamma(U,R\mathscr{H}om(Rp_{1,!}((p_2^{-1}\mathscr{F}_U)_{Z_\gamma}),\mathscr{G}))).\end{aligned}$$

Now it suffices to show that $Rp_{1,!}((p_2^{-1}\mathscr{F}_U)_{Z_\gamma})\simeq\Bbbk_\gamma\star\mathscr{F}$.

Write $\bar{p}_{1,2}=p_{1,2}|_{Z_\gamma}$. For any $\mathscr{F}\in D^b(E)$, we have a canonical morphism

$$\Bbbk_\gamma\star\mathscr{F}\to\Bbbk_\gamma\star(R\bar{p}_{2,*}\bar{p}_2^{-1}\mathscr{F})\to\Bbbk_\gamma\star(R\bar{p}_{1,*}\bar{p}_2^{-1}\mathscr{F}),$$

where the last morphism is given by the restriction $\Bbbk_\gamma\star R\bar{p}_{2,*}\to\Bbbk_\gamma\star R\bar{p}_{1,*}$ from $\bar{p}_1^{-1}(U)\to\bar{p}_2^{-1}(U)$ for any γ-invariant open subset U. We show that this gives an isomorphism. Note that $\bar{p}_1\bar{p}_2^{-1}(K)=K-\gamma$, and $\bar{p}_1:\bar{p}_2^{-1}(K)\to K-\gamma$ has proper contractible fiber if K is a closed ball. Thus by the noncharacteristic deformation lemma

$$\begin{aligned}H^jR\bar{p}_{1,*}\bar{p}_2^{-1}\mathscr{F}_x&\simeq\varinjlim_{K:x\in K}H^j(K,R\bar{p}_{1,*}\bar{p}_2^{-1}\mathscr{F})\simeq\varinjlim_{K:x\in K}H^j(\bar{p}_1^{-1}K,\bar{p}_2^{-1}\mathscr{F})\\&\simeq\varinjlim_{K:x\in K}H^j(K-\gamma,\mathscr{F})=H^j(\Bbbk_\gamma\star\mathscr{F})_x.\end{aligned}$$

This completes the proof. □

Proposition 3.13. *(1). If $f:M\to N$ is a submersion, then*

$$Rf_{d,!}f_\pi^{-1}\mu hom(\mathscr{F},\mathscr{G})\simeq\mu hom(f^!\mathscr{F},f^{-1}\mathscr{G}\otimes\omega_{N/M})\simeq\mu hom(f^{-1}\mathscr{F}\otimes\omega_{N/M},f^!\mathscr{G});$$

(2). If $f:M\to N$ is a closed embedding, then

$$Rf_{\pi,!}f_d^{-1}\mu hom(\mathscr{F},\mathscr{G})\simeq\mu hom(Rf_*\mathscr{F},Rf_!\mathscr{G})\simeq\mu hom(Rf_!\mathscr{F},Rf_*\mathscr{G}).$$

Proposition 3.14 (Sato's exact triangle)**.** *Let $\mathscr{F},\mathscr{G}\in D^b(M)$. Suppose $\mathscr{F}$ is cohomologically constructible. Then there is an exact triangle*

$$D'\mathscr{F}\otimes\mathscr{G}\to R\mathscr{H}om(\mathscr{F},\mathscr{G})\to R\dot{\pi}_*\mu hom(\mathscr{F},\mathscr{G})\xrightarrow{[1]}.$$

Proof. By Proposition 1.5 we know that $R\Gamma_\Delta R\mathscr{H}om(p_1^{-1}\mathscr{F},p_2^!\mathscr{G})\simeq R\mathscr{H}om(\mathscr{F},\mathscr{G})$. On the other hand, since $p_2:M\times M\to M$ is a submersion

$$\begin{aligned}R\mathscr{H}om(p_1^{-1}\mathscr{F},p_2^!\mathscr{G})|_\Delta\otimes\omega_{\Delta|M\times M}\simeq&R\mathscr{H}om(p_1^{-1}\mathscr{F},p_2^{-1}\mathscr{G}\otimes\omega_{M\times M|M})|_\Delta\otimes\omega_{\Delta|M\times M}\\\simeq&R\mathscr{H}om(p_1^{-1}\mathscr{F},p_2^{-1}\mathscr{G})|_\Delta.\end{aligned}$$

When $\mathscr{F}$ is cohomologically constructible, we have $R\mathscr{H}om(p_1^{-1}\mathscr{F},p_2^{-1}\mathscr{G})|_\Delta\simeq(p_1^{-1}D'\mathscr{F}\otimes p_2^{-1}\mathscr{G})|_\Delta\simeq D'\mathscr{F}\otimes\mathscr{G}$. This completes the proof. □

The following theorem follows from the singular support estimate of microlocalizations.

Theorem 3.15. *Let* $\mathscr{F}, \mathscr{G} \in D^b(M)$. *Then*

$$SS(\mu hom(\mathscr{F}, \mathscr{G})) \subset C(SS(\mathscr{F}), SS(\mathscr{G})).$$

Finally, we are able to prove the involutivity theorem for singular supports. Here being involutive just means being coisotropic via the standard symplectic structure on T^*M. Before stating the theorem, we first give a definition of being coisotropic for a possibly singular subvariety.

Definition 3.6. *Let* $S \subset T^*M$. S *is coisotropic at* $p \in S$ *if for any* $\nu \in T_p^*(T^*M)$ *such that the normal cone* $C_p(S, S) \subset \ker \nu \subset T_p(T^*M)$, $\nu \in C_p(S)$.

Note that when S is smooth near p, then this condition is just saying for any ν such that $T_pS \subset \ker \nu$, $\nu \in T_pS$. This coincides with our usual definition of being coisotropic.

Theorem 3.16. *Let* $\mathscr{F} \in D^b(M)$. *Then* $SS(\mathscr{F})$ *is coisotropic.*

Proof. Let $S = SS(\mathscr{F})$, $p \in S$ and $\nu \in T_p^*(T^*M)$ such that $C_p(S, S) \subset \ker \nu$. Suppose

$$\nu \notin C_p(S).$$

Then one can find a closed subset $Z \subset T^*M$ such that $S \subset Z$ and $\langle \nu, \lambda \rangle < 0$ for any $\lambda \in N_p^*Z \backslash \{0\}$. In fact by the assumption there is an open cone γ with vertex at p containing ν such that $\gamma \cap S = \emptyset$. We can let $S = T^*M \backslash \gamma$.

On the other hand, one should notice that since $C_p(S, S) \subset \ker \nu$, $SS(\mu hom(\mathscr{F}, \mathscr{F})) \cap T_p^*(T^*M) \subset C_p(S, S) \subset \ker \nu$. Hence

$$SS(\mu hom(\mathscr{F}, \mathscr{F})) \cap N_p^*Z \subset \{0\}.$$

This tells us that

$$R\Gamma_Z(\mu hom(\mathscr{F}, \mathscr{F}))_p \simeq 0.$$

However, since $\mathrm{supp}(\mu hom(\mathscr{F}, \mathscr{F})) \subset S$, we have $\mu hom(\mathscr{F}, \mathscr{F})_p = R\Gamma_Z(\mu hom(\mathscr{F}, \mathscr{F}))_p \simeq 0$, i.e. $p \notin SS(\mathscr{F})$. A contradiction. $\square$

3.4. **Localization of $D^b(M)$.** As we've said at the beginning, we define the localization of $D^b(M)$ along $D^b_{M \cup T^*M \backslash \Lambda}(M)$ to be $D^b(M; \Lambda)$. This means under the natural functor

$$D^b(M) \to D^b(M; \Lambda),$$

all sheaves whose singular supports are away from Λ is mapped to zero.

As we have mentioned before, it is a general phenomenon that taking localization is the same as taking direct limit. In this case, we have

$$Hom_{D^b(M;\Lambda)}(\mathscr{F}, \mathscr{G}) = \varinjlim_{\mathscr{F}' \xrightarrow{\sim} \mathscr{F} \text{ on } \Lambda} Hom_{D^b(M)}(\mathscr{F}', \mathscr{G}) = \varinjlim_{\mathscr{G} \xrightarrow{\sim} \mathscr{G}' \text{ on } \Lambda} Hom_{D^b(M)}(\mathscr{F}, \mathscr{G}').$$

Now we study its relation with the functor μhom. Recall that $Hom_{D^b(M)}(\mathscr{F}, \mathscr{G}) = H^0(T^*M, \mu hom(\mathscr{F}, \mathscr{G}))$. We have a canonical morphism

$$Hom_{D^b(M;\Lambda)}(\mathscr{F}, \mathscr{G}) \to H^0(T^*M, \mu hom(\mathscr{F}, \mathscr{G})).$$

Theorem 3.17. *Let* $p \in T^*M$, $\mathscr{F}, \mathscr{G} \in D^b(M)$. *Then*

$$Hom_{D^b(M;p)}(\mathscr{F}, \mathscr{G}) \simeq H^0(\mu hom(\mathscr{F}, \mathscr{G}))_p.$$

Proof. Write $p = (x, \xi) \in T^*M$. Consider a local Euclidean chart, by proposition 3.12 we have

$$\begin{aligned} H^0\mu hom(\mathscr{F}, \mathscr{G})_{(x,\xi)} &= \varinjlim_{U: x\in U;\, \gamma \subset (\{\xi\}^\vee)^\circ \cup \{0\}} H^0(R\Gamma(U, R\mathscr{H}om(\Bbbk_\gamma \star \mathscr{F}_U, \mathscr{G}))) \\ &= \varinjlim_{U: x\in U;\, \gamma \subset (\{\xi\}^\vee)^\circ \cup \{0\}} Hom((\Bbbk_\gamma \star \mathscr{F}_U)_U, \mathscr{G}))). \end{aligned}$$

The canonical morphism $(\Bbbk_\gamma \star \mathscr{F}_U)_U \to \mathscr{F}$ is an isomorphism at (x, ξ). Therefore the morphism $Hom_{D^b(M;p)}(\mathscr{F}, \mathscr{G}) \to H^0(\mu hom(\mathscr{F}, \mathscr{G}))_p$ is injective. For any $s \in H^0(\mu hom(\mathscr{F}, \mathscr{G}))_p$, there exists U and γ, $\bar{s} \in Hom((\Bbbk_\gamma \star \mathscr{F}_U)_U, \mathscr{G}))$ that represents s. This shows that the sequence $(\Bbbk_\gamma \star \mathscr{F}_U)_U$ is a final sequence. Hence $Hom_{D^b(M;p)}(\mathscr{F}, \mathscr{G}) \to H^0(\mu hom(\mathscr{F}, \mathscr{G}))_p$ is surjective. □

Here we study more carefully the localized category $D^b(M, p)$ and see if we can always choose a representative in $D^b(M)$ with good properties. The following proposition can be generalized to more general conical subsets by contact transformations later on.

Proposition 3.18. *Let $j : N \to M$ be a closed embedding. Let $p \in T^*_N M$ and $\mathscr{F} \in D^b(M)$.*

(1). If $SS(\mathscr{F}) \subset \pi^{-1}(N)$ in a neighbourhood of p, then there exists $\mathscr{G} \in D^b(N)$ so that $\mathscr{F} \simeq Rj_*\mathscr{G}$ in $D^b(M, p)$;

(2). If $SS(\mathscr{F}) \subset T^*_N M$ in a neighbourhood of p, then there exists $L \in D^b(\mathrm{Mod}(\Bbbk))$ so that $\mathscr{F} \simeq L_N$ in $D^b(M, p)$;

Proof. (1). By induction on the dimension of N, we may assume that N is a hypersurface defined by $\varphi = 0$. We have $\varphi(x) = 0$. Assume that $d\varphi(x) = \xi$. Let $U_\pm = \varphi^{-1}(\mathbb{R}_\pm)$ and $i_\pm : U_\pm \to M$. Then we know by Theorem 1.14 that

$$SS(Ri_{-,*}i_-^{-1}\mathscr{F}) \subset SS(i_-^{-1}\mathscr{F}) \hat{+} N^*(\mathbb{R}^n_-).$$

Hence as $SS(i_-^{-1}\mathscr{F})$ is disjoint from N, we know that $(x, \xi) \notin SS(Ri_{-,*}i_-^{-1}\mathscr{F})$. By the exact triangle

$$R\Gamma_{\overline{U}_+}(\mathscr{F}) \to \mathscr{F} \to Ri_{-,*}i_-^{-1}\mathscr{F} \xrightarrow{[1]},$$

$R\Gamma_{\overline{U}_+}(\mathscr{F}) \to \mathscr{F}$ is an isomorphism at p. Now we may assume that $\mathrm{supp}(\mathscr{F}) \subset \overline{U}_+$. On the other hand, we also know $p \notin SS(Ri_{+,*}i_+^{-1}\mathscr{F})$. Hence $\mathscr{F} \to Rj_*j^{-1}\mathscr{F}$ is an isomorphism at p. We can let $\mathscr{G} = j^{-1}\mathscr{F}$.

(2). We have $SS(\mathscr{G}) \subset N$ in a neighbourhood of p. Hence $\mathscr{G} = j^{-1}\mathscr{F} \simeq L_N$ at p by the non-characteristic deformation lemma and the fact that

$$SS(j^{-1}\mathscr{F}) = j_d j_\pi^{-1}(SS(\mathscr{F})) \subset N \subset T^*N,$$

since $j : N \to M$ has proper contractible fibers. □

4. Simple Sheaves

4.1. **Contact Transformations.** Let U, V are open subsets in T^*M, T^*N. Then for $\Lambda \subset U \times V^{\mathrm{op}}$, if $\pi_1 : \Lambda \to U$ and $\pi_2 : \Lambda \to V$ are diffeomorphisms, and Λ is a Lagrangian submanifold, then

$$\chi = (\pi_2^{\mathrm{op}}|_\Lambda) \circ (\pi_1|_\Lambda) : V \to U$$

is called a contact transformation. In particular, note that a symplectomorphism $U \xrightarrow{\sim} V$ defines a contact transformation.

Theorem 4.1. *Let $\chi : U \to V$ be a contact transformation. Then for any $x \in X, y \in Y$, there exists neighbourhoods M', N' of $\pi(x), \pi(y)$, neighbourhoods U', V' of x, y satisfying*

$$U' \subset T^*M' \cap U,\ V' \subset T^*N \cap V,$$

so that $\chi : U' \to V'$ is also a contact transformation, there exists $\Phi : D^b(M'; U') \xrightarrow{\sim} D^b(N'; V')$, and

$$\chi_* \mu hom(\mathscr{F}, \mathscr{G}) \simeq \mu hom(\Phi(\mathscr{F}), \Phi(\mathscr{G})).$$

4.2. **Pure and Simple Sheaves.** When analyzing the microlocal behaviour of the sheaf category, the simplest objects we would like to work with is those who are microlocally supported in a single degree, so that we will be under the classical setting instead of the original derived setting. Roughly speaking those are what we call pure sheaves. If in addition, the sheaf has rank one, then it is called a simple sheaf.

Our goal is to make it clear what it means by microlocally supported in a single degree or microlocally rank one.

Definition 4.1. *Let $\Lambda \subset T^*M \backslash M$ be a conical subset. $D^b_{(\Lambda)}(M)$ is the full subcategory of $D^b(M)$ in which for any $\mathscr{F}$ there is a neighbourhood U of Λ in $T^*M \backslash M$ such that $(SS(\mathscr{F}) \backslash M) \cap U \subset \Lambda$.*

The reason we want to consider this category is basically because of proposition 3.18. Basically after localization the sheaves in $D^b_{(\Lambda)}(M)$ behave well and have good representatives.

Let $\Lambda \subset T^*M$ be a conical Lagrangian, $\varphi : M \to \mathbb{R}$ a smooth function. We say φ is transverse to Λ if $\Lambda \pitchfork \Lambda_\varphi = \{(x, d\varphi(x)) | x \in M\}$. Let τ_φ be the Maslov potential.

Proposition 4.2. *Let $\Lambda = T^*_N M$ be a conical Lagrangian, $\varphi_{0,1}$ be transverse to Λ at $p = (x, \xi)$ and $\mathscr{F} \in D^b_{(\Lambda)}(M)$. Then*

$$(R\Gamma_{\varphi_1 \geq 0} \mathscr{F})_x \simeq (R\Gamma_{\varphi_0 \geq 0} \mathscr{F})_x [(\tau_{\varphi_0}(p) - \tau_{\varphi_1}(p))/2].$$

Proof. Without loss of generality, we assume that in a local chart N is defined by $x_1 = ... = x_k = 0$. The tangent space of Λ_φ is

$$T_p \Lambda_\varphi = \Big\{ (x, \xi) \Big| \xi_j = \sum_{i=1}^n \partial_i \partial_j \varphi(x) x_i \Big\}.$$

Since $T_p\Lambda_\varphi$ is transverse to $T_p\Lambda = \{(x, \xi) | x_1 = ... = x_k = \xi_{k+1} = ... = \xi_n = 0\}$, we know $(\partial_i \partial_j \varphi(x))_{k+1 \leq i,j \leq n}$ is non-degenerate. By Morse lemma, one may assume that

$$\varphi|_N = \sum_{j=k+1}^n a_j x_j^2,\ \ a_{k+1}, ..., a_{k+l} < 0,\ a_{k+l+1}, ..., a_n > 0.$$

The corresponding Maslov potential at p is $\tau_\varphi(p) = -\mathrm{sgn}(D^2\varphi|_N) = 2l + k - n$. Therefore we have

$$(R\Gamma_{\varphi \geq 0} \mathscr{F})_x [\tau_\varphi(p)/2] \simeq (R\Gamma_\varphi L_N)_0 [l + (k-n)/2] = L[(k-n)/2]$$

which is independent of the choice of φ. □

Proposition 4.3. *Let $p_0 \in T^*M_0, p_1 \in T^*M_1$, and $\chi : U_0 \xrightarrow{\sim} U_1$ is a contact transformation between neighbourhoods of p_0 and p_1. Suppose φ_0, φ_1 are smooth functions so that $p_0 \in \Lambda_{\varphi_0}, p_1 \in \Lambda_{\varphi_1}$, and $\chi(T^*_{\varphi_0^{-1}(0)} M_0) = T^*_{\varphi_1^{-1}(0)} M_1$. Then for any Lagrangian plane $l \subset T_p(T^*M)$,*

$$\begin{aligned} R\Gamma_{\varphi_0 \geq 0}(\mathscr{F})_{x_0} \simeq R\Gamma_{\varphi_1 \geq 0}(\chi_* \mathscr{F})_{x_1} [(\tau_{\varphi_1}(l) - \tau_{\varphi_0}(l))/2 + (n-1)/2 \\ + \tau(T_{p_0} T^*_{x_0} M_0, l, \chi^{-1}(T_{p_1} T^*_{x_1} M_1))/2]. \end{aligned}$$

Definition 4.2. *Let $\Lambda \subset T^*M$ be a conical Lagrangian. $\mathscr{F} \in D^b_{(\Lambda)}(M)$ is called a pure sheaf if for some $p = (x, \xi) \in \Lambda$, $(R\Gamma_{\varphi_1 \geq 0}\mathscr{F})_x$ is concentrated in a single degree. It is called a simple sheaf if $(R\Gamma_{\varphi_1 \geq 0}\mathscr{F})_x$ is rank one.*

The following proposition explains what is the relationship between pure/simple sheaves and microlocalization. It tells us that the condition in the definition is indeed a microlocal condition. The propositions can be applied to general conical Lagrangians using contact transformations.

Proposition 4.4. *Let $N \subset M$ be a submanifold, $\Lambda = T^*_N M$ and $\mathscr{F}, \mathscr{G} \in D^b_{(\Lambda)}(M)$. For $p = (x, \xi) \in \Lambda$, $\varphi : M \to \mathbb{R}$ such that $\varphi(x) = 0, d\varphi(x) = \xi$,*

$$\mu hom(\mathscr{F}, \mathscr{G})_p = RHom(R\Gamma_{\varphi \geq 0}(\mathscr{F})_x, R\Gamma_{\varphi \geq 0}(\mathscr{G})_x).$$

Proof. By proposition 3.18, one only needs to consider the case when $\mathscr{F} \simeq K_N, \mathscr{G} \simeq L_N$ at p. Then on the left hand side

$$\mu hom(\mathscr{F}, \mathscr{G})_p = \mu hom(K_N, L_N)_p = RHom(K, L).$$

On the right hand side, if $k = \dim N$, then

$$RHom(R\Gamma_{\varphi \geq 0}(\mathscr{F})_p, R\Gamma_{\varphi \geq 0}(\mathscr{G})_p) = RHom(K, L).$$

Therefore we are done. □

Proposition 4.5. *Let $\Lambda = T^*_N M$. $\mathscr{F} \in D^b_{(\Lambda)}(M)$ is pure iff $\mu hom(\mathscr{F}, \mathscr{F})|_\Lambda$ is concentrated in degree zero; it is simple iff $\mu hom(\mathscr{F}, \mathscr{F})|_\Lambda \simeq \Bbbk_\Lambda$.*

Proof. Again pick $K, L \in D^b(\mathrm{Mod}(\Bbbk))$ so that $K_N \simeq \mathscr{F}, L_N \simeq \mathscr{G}$ at $p \in \Lambda$. Then $\mu hom(\mathscr{F}, \mathscr{G})_p = RHom(K, L)$ and the previous proposition tells us that it does not depend on p. Then the result follows from linear algebra. □

4.3. **Derived Category μsh_Λ.** The functor $\Lambda \mapsto D^b(M, \Lambda)$ defines a presheaf of categories whose stalk is $D^b(M, p)$ (where morphisms are given by the stalk of μhom). Now we illustrate why simple sheaves play an important role in microlocal sheaf theory. Basically the reason is that a simple sheaf gives a framing that identifies the sheafification of categories with derived local systems.

Definition 4.3. *Let $\Lambda \subset T^*M$ be a conical subset. The presheaf of categories (prestack) μsh^0_Λ is*

$$\Lambda_0 \mapsto D^b(M, \Lambda_0), \;\; \Lambda_0 \subset \Lambda.$$

The sheafification is μsh_Λ.

Theorem 4.6. *Let $\Lambda \subset T^*M$ be a closed conical Lagrangian. Suppose $\mathscr{F} \in D^b_{(\Lambda)}(M)$ is a simple sheaf. Then there is an equivalence of categories*

$$\mu sh_\Lambda(\Lambda) \to D^b Loc(\Lambda); \;\; \mathscr{G} \mapsto \mu hom(\overline{\mathscr{F}}, \mathscr{G})|_\Lambda.$$

By discussions in the previous subsection, it suffices to show the following:

Proposition 4.7. *Let $\Lambda \subset T^*M$ be a closed conical Lagrangian and $p = (x, \xi) \in \Lambda$. Then there exists a neighbourhood $\Lambda_0 \subset \Lambda$ of p such that*

(1). there exists $\mathscr{F} \in D^b_{(\Lambda_0)}(M)$ that is simple along Λ_0;

(2). for any $\mathscr{G} \in D^b_{(\Lambda_0)}(M)$ there exists a neighbourhood U of Λ_0 such that $\mathscr{F} \otimes^L L_M \xrightarrow{\sim} \mathscr{G}$ in $D^b(M, U)$, where $L = \mu hom(\mathscr{F}, \mathscr{G})_p$.

Proof. Without loss of generality, we may assume that $\Lambda = T^{*,+}_N M$ where $N \subset M$ is a hypersurface. In a local coordinate chart, $\Lambda = \{(x,\xi) \in T^*\mathbb{R}^n | x_1 = 0, \xi_1 > 0, \xi_2 = ... = \xi_n = 0\}$, $N = 0 \times \mathbb{R}^{n-1}$. Then we can define the simple sheaf

$$\mathscr{F} = \Bbbk_{[0,+\infty)\times\mathbb{R}^{n-1}}.$$

For any $\mathscr{G} \in D^b_{(\Lambda)}(M)$, there exists $K \in D^b(\mathrm{Mod}(\Bbbk))$ such that $\mathscr{G}_{(-1,0,...,0)} = K$. We have a long exact sequence

$$\mathscr{G}_{(1,0,...,0)} \to \mathscr{G}_{(-1,0,...,0)} \to R\Gamma_{[0,+\infty)\times\mathbb{R}^{n-1}}(\mathscr{G})_{(0,0,...,0)} \xrightarrow{[1]} .$$

Therefore if $L = R\Gamma_{[0,+\infty)\times\mathbb{R}^{n-1}}(\mathscr{G})_{(0,0,...,0)}$ then in $D^b(M,\Lambda)$ since $K_{\mathbb{R}^n} \simeq 0$ we have

$$\mathscr{G} \simeq Cone(L_{[0,+\infty)\times\mathbb{R}^{n-1}}[-1] \to K_{\mathbb{R}^n}) \simeq L_{[0,+\infty)\times\mathbb{R}^{n-1}}.$$

This completes the proof of the proposition. □

5. Quantization of Hamiltonian Isotopies

We've already seen that singular supports of sheaves are coisotropic subsets, which naturally arise as objects in symplectic geometry. In this section we show that indeed the category of sheaves with given singular support $D^b_\Lambda(M)$ is an invariant of Λ, in other words they are invariant under Hamiltonian isotopy.

Here since we don't care about the zero section, we use the following conventions throughout the section: $\dot{T}^*M = T^*M\backslash M$, $\dot{SS}(\mathscr{F}) = SS(\mathscr{F})\backslash M$, $D^b_\Lambda(M) = D^b_{\Lambda\cup M}(M)$ and $D^b(M;\Lambda) = D^b(M)/D^b_{(T^*M\backslash\Lambda)\cup M}(M)$. Now we introduce the main theorem in this section.

Theorem 5.1 (Guillermou-Kashiwara-Schapira)**.** *Let $\varphi_t\,(t \in I)$ be a homogeneous Hamiltonian isotopy of $\dot{T}^*M$ such that $\varphi_0 = \mathrm{id}$, and $\Lambda_t\,(t \in I)$ be the graph of $\varphi_t\,(t \in I)$ in $\dot{T}^*(M \times M \times I)$. Then up to isomorphism, there exists a unique sheaf $\mathscr{K} \in D^{lb}(M \times M \times I)$ such that*

(1). $\dot{SS}(\mathscr{K}) \subset \Lambda$;

(2). $\mathscr{K}|_{t=0} = \Bbbk_\Delta$;

Let $\mathscr{K}_{-t} = (a \times \mathrm{id}_I)^{-1}R\mathscr{H}om(\mathscr{K}, \omega_M \boxtimes \Bbbk_M \boxtimes \Bbbk_I)$ (where $a(x,y) = (y,x)$) and $\mathscr{K} \circ \mathscr{L} = R\pi_{13,*}(\pi_{12}^{-1}\mathscr{K} \otimes \pi_{23}^{-1}\mathscr{L})$. Then $\mathscr{K}$ in addition satisfies

(3). $\pi_{1,2} : \mathrm{supp}(\mathscr{K}) \to M \times I$ are proper;

(4). $\mathscr{K}_t \circ \mathscr{K}_{-t} \simeq \mathscr{K}_{-t} \circ \mathscr{K}_t \simeq \Bbbk_\Delta$;

(5). If $\varphi_t|_U \equiv \mathrm{id}_U$, then $\mathscr{K}|_{(U\times M\cup M\times U)\times I} = \Bbbk_{(\Delta\cap(U\times M\cup M\times U))\times I}$.

Note that in particular (4) implies that

$$\mathscr{K}_t : D^{lb}(M) \to D^{lb}(M)$$

is an equivalence of categories.

5.1. Uniqueness.

We first show uniqueness of $\mathscr{K}$. Let's write

$$B = \{(x,y,t)|(\{(x,y)\} \times [0,t]) \cap \dot{\pi}(\Lambda) \neq \emptyset\}.$$

We will show that for a sheaf $\mathscr{K}$ satisfying (1) and (2), it will satisfy (3)-(5) and is unique.

Proof of Uniqueness. First we prove (3). In fact we show that $\mathrm{supp}(\mathscr{K}) \subset B$. Otherwise suppose for $(x,y,t) \notin B$, there is a neighbourhood $(U \times V \times J) \cap \pi(\Lambda) = \emptyset$. Then $\mathscr{K}|_{U\times V\times J}$ is constant. Note that by definition of B, $0 \in J$. Since $\mathscr{K}|_{t=0} = \Bbbk_\Delta$, $(x,y) \notin \Delta$. This shows that $\mathscr{K}|_{U\times V\times J} = 0$. Now the claim follows from the fact that $\pi_{1,2} : B \to M \times I$ are proper.

Next we prove (4). It suffices to show that

$$\mathscr{K} \circ_I \mathscr{K}^{-1} = R\pi_{13,*}(\pi_{12}^{-1}\mathscr{K}_t \otimes \pi_{23}^{-1}\mathscr{K}_{-t}) \simeq \Bbbk_{\Delta\times I}.$$

We estimate the singular support of $\mathscr{K} \circ_I \mathscr{K}^{-1}$. In fact, $\dot{S}S(\mathscr{K}) \subset \Lambda$, so $\dot{S}S(\mathscr{K}^{-1}) \subset a(\Lambda)$ where $a(x,\xi,y,\eta,t,\tau) = (y,-\eta,x,-\xi,t,-\tau)$. This tells us that

$$\dot{S}S(\mathscr{K} \circ_I \mathscr{K}^{-1}) \subset \dot{T}^*(M \times M) \times I.$$

Hence $\mathscr{K} \circ_I \mathscr{K}^{-1}$ is constant along I, which means $(\mathscr{K} \circ_I \mathscr{K}^{-1}|_t) \equiv \Bbbk_\Delta$.

Then we show (5). This is because $\Lambda \cap (U \times M \cup M \times U) \times I = (\Delta \cap (U \times M \cup M \times U)) \times I$, which tells us that

$$\dot{S}S(\mathscr{F}) \subset T^*(U \times M \cup M \times U) \times I,$$

which is to say $\mathscr{K}|_{(U\times M\cup M\times U)\times\{t\}} \equiv \Bbbk_\Delta$.

Finally we show uniqueness up to isomorphisms. Suppose one can find $\mathscr{K}_0, \mathscr{K}_1$ satisfying (1) and (2), then there is a unique isomorphism

$$\psi : \mathscr{K}_0 \to \mathscr{K}_1$$

so that $i_0^{-1}\psi : \Bbbk_\Delta \to \Bbbk_\Delta$ is the identity. Let $\mathscr{L} = \mathscr{K}_0 \circ_I \mathscr{K}_1^{-1}$. By the same argument as we prove (4) we know that $\dot{S}S(\mathscr{F}) \subset \dot{T}^*(M \times M) \times I$, so $\mathscr{L} \simeq \Bbbk_{\Delta\times I}$.

$$\mathscr{K}_0 \simeq \mathscr{L} \circ_I \mathscr{K}_1 \simeq \mathscr{K}_1.$$

This completes the proof. □

5.2. **Existence.** In this section we show that there exists a sheaf $\mathscr{K} \in D^{lb}(M \times M \times I)$ that satisfies (1) and (2). This will prove our theorem.

The issue is that essentially the only sheaf quantization we are only able to construct is the constant sheaf on a submanifold $N \subset M$. When $\Lambda = T^*_N M$ is a conormal bundle of a submanifold, we may define $\mathscr{K} = \Bbbk_N$. Unfortunately this is not always the case. However this is also not that complicated, since we know the front projection of a conical Lagrangian (or a Legendrian) is generically a hypersurface. Suppose the hypersurface is separating M as $U_+ \cup U_-$. Then one may be able to work with the sheaf $\Bbbk_{U_-}$.

The question is how do we deform. The following lemma gives a way, which is essentially to deform by the geodesic flow. Intuitively, when you run the geodesic flow, any point (corresponding to a cotangent fiber in the cotangent bundle) will expand to a circle (corresponding to its inward conormal).

Lemma 5.2. *Let Ω be a neighbourhood of $\Delta \subset M \times M$, $f \in C^\infty(M)$ be such that*

(a). $f|_\Delta \equiv 0$;

(b). $f(x,y) > 0$ for $(x,y) \in \Omega\backslash\Delta$;

(c). $D_iD_jf(x,x)$ is positive definite for $(x,x) \in \Delta$.

Then for a relatively compact subset $U \subset M$, there exists $\epsilon_0 > 0$ and $\Omega_0 \subset M \times M$ such that

(1). $\Delta \subset \Omega_0 \subset \Omega \cap (M \times U)$;

(2). For $Z_{\epsilon_0} = \{(x,y) \in \Omega_0 | f(x,y) \le \epsilon_0\}$, $\pi_2 : Z_{\epsilon_0} \to U$ is proper;

(3). For $y \in U, \epsilon \in [0,\epsilon_0]$, $\{x \in M | (x,y) \in \Omega_0, f(x,y) < \epsilon\} \simeq \mathbb{R}^n$;

(4). $D_xf(x,y) \ne 0, D_yf(x,y) \ne 0$ for $x,y \in \Omega_0\backslash\Delta$;

(5). Let $T^*_{\partial Z_{\epsilon_0},-}\Omega_0$ be the inward conormal bundle, $\pi_2 : T^*_{\partial Z_{\epsilon_0},-}\Omega_0 \xrightarrow{\sim} \dot{T}^*U$ and $\pi_1 : T^*_{\partial Z_{\epsilon_0},-}\Omega_0 \xrightarrow{\sim} \dot{T}^*M$ is an open embedding.

Let $\mathscr{L} = \Bbbk_{Z_{\epsilon_0}} \in D^{lb}(M \times U)$. Then $\dot{S}S(\mathscr{L}) \subset T^*_{\partial Z_{\epsilon_0},-}\Omega_0$ and $\mathscr{L}^{-1} \circ \mathscr{L} \simeq \Bbbk_{\Delta_U}$.

Proof. Condition (1)-(5) are easy to be satisfied. It suffices to check the last assertions. $\dot{S}S(\mathscr{L}) \subset T^*_{\partial Z_{\epsilon_0},-}\Omega_0$ is essentially by definition. In addition,

$$\dot{S}S(\mathscr{L}^{-1} \circ \mathscr{L}) \subset \dot{T}^*_{\Delta_U}(U \times U).$$

Note that because $\mathscr{L}$ is simple, there is a canonical morphism $\mathscr{L}^{-1} \circ \mathscr{L} \to \Bbbk_{\Delta_U}$ which is an isomorphism along $\dot{T}^*_{\Delta_U}(U \times U)$. Thus $Cone(\mathscr{L}^{-1} \circ \mathscr{L} \to \Bbbk_{\Delta_U})$ is a local system. In addition, (write $Z = Z_{\epsilon_0}$)

$$\begin{aligned}\delta^{-1}(\mathscr{L}^{-1} \circ \mathscr{L}) &\simeq R\pi_{2,!}(\mathscr{L} \otimes R\mathscr{H}om(\mathscr{L}, \Bbbk_{M\times U}) \otimes \pi_2^! \Bbbk_U) \\ &\simeq R\pi_{2,!}(\Bbbk_Z \otimes \Bbbk_{Z^\circ} \otimes \pi_2^! \Bbbk_U) \simeq \Bbbk_U\end{aligned}$$

since the fiber of $\pi_2 : Z^\circ \to U$ is $\mathbb{R}^n$. Hence $\delta^{-1}(\mathscr{L}^{-1} \circ \mathscr{L}) = \Bbbk_{\Delta_U}$, which means $\delta^{-1}Cone(\mathscr{L}^{-1} \circ \mathscr{L} \to \Bbbk_{\Delta_U}) = 0$. Since it is locally constant, we can conclude that it is 0. □

Now we try to illustrate why in the case when the front projection of Λ is a hypersurface, it is easier to extend the sheaf by the Hamiltonian isotopy. In fact we are just extending $Z \subset M \times M$ to $\tilde{Z} \subset M \times M \times (-\epsilon, \epsilon)$.

Lemma 5.3. *Let $Z \subset M \times M$ be an open subset with smooth boundary. $\Lambda \subset \dot{T}^*(M \times M \times I)$ be a closed conical Lagrangian and $\Lambda_t = \Lambda \circ T^*_t I$. We assume that*

(1). $\Lambda|_{t=0} = \dot{SS}(\Bbbk_Z)$;

(2). $\Lambda \cap \dot{T}^*(((M \times M)\backslash K) \times I) = (\Lambda_0 \backslash K) \times I$;

(3). $\Lambda \to \dot{T}^*(M \times M) \times I$ is a closed embedding.

Then there exists $\epsilon > 0$ and an closed subset $\tilde{Z} \subset M \times M \times (-\epsilon, \epsilon)$ such that

(1). $\tilde{Z} \cap (M \times M \times \{0\}) = Z$;

(2). $\Lambda = \dot{SS}(\Bbbk_{\tilde{Z}})$.

Proof of existence. Assume that $\varphi_t\,(t \in I)$ is compactly supported on N. Choose a relative compact open subset $N \subset U \subset M$, and apply Proposition 5.2. Then $\mathscr{L} = \Bbbk_{Z_{\epsilon_0}} \in D^b(M \times U)$, $\dot{SS}(\mathscr{L}) \subset T^*_{\partial Z_{\epsilon_0},-}(M \times U)$ and $\mathscr{L}^{-1} \circ \mathscr{L} = \Bbbk_{\Delta_U}$. Write

$$\tilde{\Lambda} = T^*_{\partial Z_{\epsilon_0},-}(M \times U) \circ \Lambda.$$

Now we apply Lemma 5.3 to $\mathscr{L} = \Bbbk_{Z_{\epsilon_0}}$, and deduce that there exists $\tilde{Z} \subset M \times U \times (-\epsilon, \epsilon)$, $\tilde{\mathscr{L}} = \Bbbk_{\tilde{Z}}$ such that

(1). $\tilde{\mathscr{L}}|_{t=0} = \mathscr{L}$;

(2). $\dot{SS}(\tilde{L}) \subset (\tilde{\Lambda} \times_I (-\epsilon, \epsilon))$;

(3). $\pi_2 : M \times U \times (-\epsilon, \epsilon) \to U \times (-\epsilon, \epsilon)$ is proper on $\operatorname{supp}(\tilde{\mathscr{L}})$.

We define the sheaf quantization

$$\mathscr{K} = \mathscr{L}^{-1} \circ_I \tilde{\mathscr{L}} \in D^b(U \times U \times (-\epsilon, \epsilon)).$$

Then by the proof in the uniqueness part, $\mathscr{K}|_{((U\times U)\backslash(N\times N))\times(-\epsilon,\epsilon)} = \Bbbk_{\Delta_{U\backslash N}\times(-\epsilon,\epsilon)}$, so it can be extended to $\mathscr{K} \in D^{lb}(M \times M \times (-\epsilon, \epsilon))$.

Now we can glue the sheaves on $M \times M \times (t_i, t_{i+1})$ as when we solve ordinary differential equations. Suppose J is the maximal interval where $\mathscr{K}$ can be defined, then since it has to be both open and closed, we are done.

In general, if $\varphi_t\,(t \in I)$ is not compactly supported, we just using an exhausting sequence of compact subsets $\{N_n\}_{n\geq 0}$ of M and cut-off the Hamiltonian isotopy outside N_n. Inductively this defines a sheaf quantization globally. □

5.3. **Topological Applications.** First of all we conclude that the sheaf category with given singular support in indeed a Legendrian invariant.

Corollary 5.4. *Let $\varphi_t\,(t \in I)$ be a homogeneous Hamiltonian isotopy of $\dot{T}^*M$ such that $\varphi_0 = \mathrm{id}$, and $\Lambda_t\,(t \in I)$ be the graph of $\varphi_t\,(t \in I)$ in $\dot{T}^*(M \times M \times I)$. Suppose $S_0 \subset \dot{T}^*M$ is a conical Lagrangian, $S = \Lambda \circ_I S_0$ and $S_t = \Lambda_t \circ S_0$. Then*

$$i_t^{-1} : D^{lb}_S(M \times I) \to D^{lb}_{S_t}(M)$$

is an equivalence.

Next we summarize some results on non-displaceability problems in the cotangent bundle. We start with the homogeneous case.

Proposition 5.5. *Let $\varphi_t\,(t \in I)$ be a Hamiltonian isotopy on T^*M, $f \in C^\infty(M)$ be such that $df(x) \neq 0$, and $\mathscr{F}_0 \in D^b(M)$ be with compact support. Assume that $R\Gamma(M, \mathscr{F}_0) \neq 0$. Then $\varphi_t(\dot{SS}(\mathscr{F}_0)) \cap \Lambda_f \neq \emptyset$. Let $S_0 = \dot{SS}(\mathscr{F}_0) \subset T^*M$ be a conical Lagrangian. Assume $\mathscr{F}_0$ is simple, $\Lambda_f \pitchfork \varphi_t(S_0)$ and the intersection is finite. Then*

$$|\varphi_t(S_0) \cap \Lambda_f| \geq \sum_{j \in \mathbb{Z}} \dim H^j(M, \mathscr{F}_0).$$

Proof. Let $\mathscr{F} = \Phi_{\mathscr{K}}(\mathscr{F}_0)$ and $\mathscr{F}_t = \Phi_{\mathscr{K}_t}(\mathscr{F}_0)$. Then $\dot{SS}(\mathscr{F}_t) \subset \varphi_t(\dot{SS}(\mathscr{F}_0))$. The first result follows from the microlocal Morse lemma.

Note that when f is simple, at $(x_0, \xi_0) \in \varphi_t(S_t) \cap \Lambda_f$,

$$\sum_{j \in \mathbb{Z}} \dim H^j(R\Gamma_{f(x) \geq f(x_0)}(\mathscr{F}_t)_{x_0}) = 1.$$

Let $\varphi_t(S_0) \cap \Lambda_f = \{(x_i, \xi_i) | i \in I\}$. This tells us that

$$|\varphi_t(S_0) \cap \Lambda_f| = \sum_{i \in I} \sum_{j \in \mathbb{Z}} \dim H^j(R\Gamma_{f(x) \geq f(x_i)}(\mathscr{F}_t)_{x_i}).$$

Now the proposition follows from the microlocal Morse inequality. □

In order to discuss non-homogeneous problems, we lift T^*M to $T^*(M \times \mathbb{R})$ and modify the non-homogeneous problem to a homogeneous problem.

Theorem 5.6 (Floer)**.** *Let $\varphi_t\,(t \in I)$ be a (not necessarily homogeneous) Hamiltonian isotopy on T^*M and there exists $K \subset T^*M$ such that $\varphi_t|_{T^*M \setminus K} \equiv \mathrm{id}$. Then $\varphi_t(M) \cap M \neq \emptyset$, and when $\varphi_t(M) \pitchfork M$,*

$$|\varphi_t(M) \cap M| \geq \sum_{i=0}^{n} \dim H^i(M; \Bbbk_M).$$

Proof. Lift $\varphi_t\,(t \in I)$ to a homogeneous Hamiltonian isotopy on $T^*(M \times \mathbb{R})$. If φ_t is defined by H_t, then write

$$\rho : T^*M \times \dot{T}^*\mathbb{R} \to T^*M;\ (x, \xi, s, \sigma) \mapsto (x, \xi/\sigma)$$

and let $d\tilde{H}_t = \sigma\rho^* dH_t + (H_t \circ \rho)d\sigma$.

Let $f = s$, $\mathscr{F}_0 = \Bbbk_M \in D^b(M \times \mathbb{R})$ and apply the previous proposition, we get the estimate for $\tilde{\varphi}_t(\dot{T}^*_M(M \times \mathbb{R})) \cap \Lambda_f$. Now let $\Sigma_t = \{(\sigma\varphi_t(x, 0), \sigma) | x \in M, \sigma \in \mathbb{R}^\times\}$. We first of all have

$$\Sigma_t \cap (M \times \{1\}) = \varphi_t(M) \cap M.$$

Next note that $\tilde{\varphi}_t(\dot{T}^*_M(M \times \mathbb{R})) \xrightarrow{\sim} \Sigma_t$ under the map $p : T^*M \times \dot{T}^*\mathbb{R} \to T^*M \times \mathbb{R}^\times$; $(x, \xi, s, \sigma) \mapsto (x, \xi, \sigma)$ because

$$\tilde{\varphi}_t(\dot{T}^*_M(M \times \mathbb{R})) = \{(\sigma\varphi_t(x, 0), u(x, 0, t), \sigma) | x \in M, \sigma \in \mathbb{R}^\times\}$$

where u is the function such that

$$\tilde{\varphi}_t(x,\xi,s,\sigma) = (x',\xi',s+u(x,\xi/\sigma,t),\sigma),\ \ \varphi_t(x,\xi/\sigma) = (x',\xi'/\sigma).$$

On the other hand we know that $p: \Lambda_f \xrightarrow{\sim} M \times \{1\}$. Therefore

$$\tilde{\varphi}_t(\dot{T}^*_M(M\times\mathbb{R})) \cap \Lambda_f \simeq \Sigma_t \cap (M\times\{1\}),$$

which finishes the proof of the theorem. □

6. Quantization of Lagrangian Submanifolds

We can use microlocal sheaf theory to study the geometry of Lagrangian submanifolds. However, sheaf theory can only detect conical Lagrangians in cotangent bundles, or equivalently, Legendrians in unit cotangent bundles. Therefore we lift a Lagrangian $L \subset T^*M$ to a Legendrian in $T^{*,\infty}_{\tau\geq 0}(M\times\mathbb{R})$. For such a lifting to exist, we require that L is exact, i.e. $\lambda_{\mathrm{std}}|_L = df_L$.

Definition 6.1. *Let $L \subset T^*M$ be an exact Lagrangian such that $\lambda_{std}|_L = df_L$. Then the Legendrian lift of L is*

$$\hat{L} = \{(x,\xi;t,+\infty)|(x,\xi)\in L,\, t = f_L(x,\xi)\} \subset T^{*,\infty}_{\tau\geq 0}(M\times\mathbb{R}).$$

The conification of L is

$$C(L) = \{(x,\xi;t,\tau)|\tau\in(0,+\infty),\,(x,\xi/\tau)\in L,\, t = f_L(x,\xi/\tau)\} \subset T^*_{\tau\geq 0}(M\times\mathbb{R}).$$

Here since we don't care about the zero section, we use the following conventions throughout the section: $\dot{T}^*M = T^*M\backslash M$, $\dot{SS}(\mathscr{F}) = SS(\mathscr{F})\backslash M$, $D^b_\Lambda(M) = D^b_{\Lambda\cup M}(M)$ and $D^b(M;\Lambda) = D^b(M)/D^b_{(T^*M\backslash\Lambda)\cup M}(M)$. Now we introduce the main theorem in this section.

Theorem 6.1 (Guillermou)**.** *Let $\Lambda \subset T^*(M\times\mathbb{R})$ be a conification of a compact embedded exact Lagrangian submanifold $L \subset T^*M$. Then there exists a sheaf $\mathscr{F} \in D^b_\Lambda(M\times\mathbb{R})$ such that $\dot{SS}(\mathscr{F}) = \Lambda$.*

Throughout this section, we will also assume that all categories are homotopy categories with dg enhancements (instead of simply triangulated categories) and all sheaves of categories are defined in the dg sense. This is because the presheaf of sheaves (when working with triangulated categories)

$$sh^{\mathrm{pre}}_\Lambda : U \mapsto D^b_{\Lambda\cap T^*U}(U)$$

is not a sheaf, but (when working with homotopy categories with dg enhancements)

$$sh^{\mathrm{pre}}_\Lambda : U \mapsto D^b_{\Lambda\cap T^*U}(U)$$

is itself a sheaf, and this will bring much convenience to us when gluing sheaves on small open subsets.

6.1. **Local Construction.** Given a Lagrangian submanifold $\Lambda \subset \dot{T}^*M$, without loss of generality we consider a simple case where the front projection $\pi|_\Lambda : \Lambda/\mathbb{R}_+ \to M$ has finite fibers.

Lemma 6.2. *Let $\Lambda \subset \dot{T}^*M$ be a conical Lagrangian so that the front projection $\pi|_\Lambda$: $\Lambda/\mathbb{R}_+ \to M$ has finite fibers. Let $p = (x,\xi) \in \dot{T}^*M$. Then there is a neighbourhood $U \subset M$ of x and $\mathscr{F} \in D^b(U)$ so that $\dot{SS}(\mathscr{F}) \subset \Lambda \cap \dot{T}^*U$ and $\mathscr{F}$ is simple along $\Lambda \cap \dot{T}^*U$.*

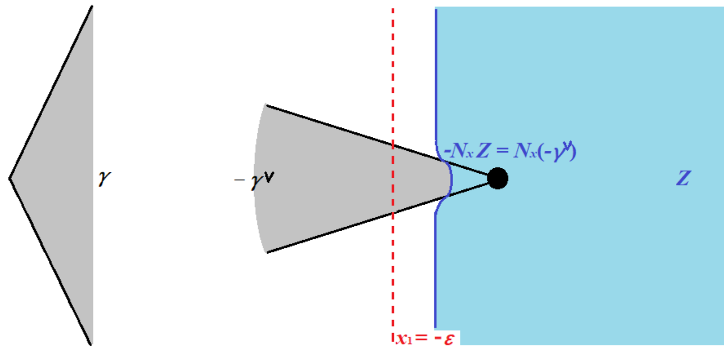

FIGURE 6. The closed cone $\gamma^\vee$ and the closed subset Z in $M = \mathbb{R}^n$.

The idea is very simple. By Proposition 4.7 one knows that there exists a sheaf $\mathscr{F}_0 \in D^b(M)$ simple at $p \in T^*M$. However that only creates a neighbourhood V around p where $\mathscr{F}_0$ is simple. In the lemma we instead want a neighbourhood around T_x^*M where $\mathscr{F}$ is simple. Hence the main difficulty is to cut off the simple sheaf so that at the fiber T_x^*M it has no singular support away from $T_x^*M \cap V$.

Proof. Without loss of genrality, we may assume that $\Lambda \cap T_x^*M = \mathbb{R}_+\xi$. By Proposition 4.7 there is a neighbourhood $V \subset T^*M$ of p, $\mathscr{F} \in D^b(M)$ such that

$$SS(\mathscr{F}_0) \cap V \subset \Lambda \cap V$$

and $\mathscr{F}_0$ is simple along Λ at p. We may assume that $T_x^*M \cap V \cap \Lambda = \mathbb{R}_+\xi$. Now pick an open convex cone $\gamma \subset T_x^*M$ such that $\xi \in \gamma$ and $\bar{\gamma} \subset T_x^*M \cap V$. Then we claim that there exists an exact triangle

$$\mathscr{F} \xrightarrow{u} \mathscr{F}_0 \to \mathscr{G} \xrightarrow{[1]},$$

so that $SS(\mathscr{G}) \cap \gamma = \emptyset$ and $\dot{T}_x^*M \cap SS(\mathscr{F}) \subset \gamma$.

Under the assumption of the claim, consider a conical neighbourhood V_γ of γ in $V \subset T^*M$. Now $\Lambda \backslash V_\gamma$ is disjoint from p, by the assumption that $\Lambda \cap T_x^*M = \mathbb{R}_+\xi$, we know there is a neighbourhood $U \subset M$ of x so that $\Lambda \cap T^*U \subset V_\gamma$. Then

$$p \notin (\dot{SS}(\mathscr{F}) \cap \dot{T}^*U) \backslash V_\gamma = (\dot{SS}(\mathscr{F}) \cap \dot{T}^*U) \backslash \Lambda.$$

We can now define $U' = M \backslash \dot{\pi}((\dot{SS}(\mathscr{F}) \cap \dot{T}^*U) \backslash \Lambda)$.

Finally it suffices to check the claim. Choose a local chart U and assume $M = \mathbb{R}^n$. Without loss of generality we assume $\gamma^\vee \subset \{x | x_1 < 0\} \subset M$. Now we pick a closed subset $Z \subset M$ so that (see Figure 6)

$$(1). \, 0 \in Z^\circ; \;\; (2). \, Z \subset \{x | x_1 \geq -\epsilon\}; \;\; (3). \, -N_x^*Z = N_x^*(\gamma^\vee).$$

Let $s : M \times M \to M$, $(x, y) \mapsto x - y$. We now define

$$\mathscr{F} = \Bbbk_{\gamma^\vee} \star R\Gamma_Z(\mathscr{F}_0) = R\pi_{2,*}(s^{-1}\Bbbk_{\gamma^\vee} \otimes \pi_1^{-1} R\Gamma_Z(\mathscr{F}_0)).$$

Let $u : \mathscr{F} \to \mathscr{F}_0$ be the composition

$$\mathscr{F} \xrightarrow{\sim} \Bbbk_{\gamma^\vee} \star R\Gamma_Z(\mathscr{F}_0) \to \Bbbk_0 \star R\Gamma_Z(\mathscr{F}_0) \xrightarrow{\sim} R\Gamma_Z(\mathscr{F}_0) \to \mathscr{F}_0.$$

Using the microlocal cutoff lemma 1.9, u is an isomorphism on $Z^\circ \times \gamma^\circ$. At the same time,

$$SS(\mathscr{F}) \subset M \times \gamma, \;\; T_0^*M \cap SS(\mathscr{F}) \subset \gamma.$$

Hence it suffices to check that u is also an isomorphism on $\partial\gamma \subset T_0^*M$.

Let $\Gamma = \{\xi \in \gamma\backslash\{0\} | \exists\, x \in U, (x,\xi) \in SS(\mathscr{F}_0)\}$. By refining U we may assume that $\Gamma \subset \gamma$. Now it suffices to show that

$$\xi \in \Gamma,\ \forall\,(0,\xi) \in SS(\mathscr{F}),\, \xi \in \partial\gamma\backslash\{0\}.$$

This will tell us that $\xi \in \gamma$. By the tensor product formula (since $\pi_2 : s^{-1}(\gamma)\cap\pi_1^{-1}(Z) \to M$ is proper), if $(0,\xi) \in SS(\mathscr{F})$, then there exists

$$(x,\xi) \in (-SS(\Bbbk_{\gamma^\vee})) \cap SS(R\Gamma_Z(\mathscr{F})).$$

Now $x \in U$, so it suffices to check for any $(x,\xi) \in (-SS(\Bbbk_\gamma)) \cap SS(R\Gamma_Z(\mathscr{F}_0))$,

$$(x,\xi) \in SS(\mathscr{F}_0).$$

Since $(x,\xi) \in (-SS(\Bbbk_{\gamma^\vee})) \cap SS(R\Gamma_Z(\mathscr{F}))$, $x \in \partial\gamma$. If $x \in Z^\circ$, $\mathscr{F}_0 \simeq R\Gamma_Z(\mathscr{F}_0)$, so $(x,\xi) \in SS(\mathscr{F})$. If $x \in \partial Z$, $N_x^*Z = N_x^*\gamma^{\mathrm{op}} = \mathbb{R}_{\geq 0}\xi$. Suppose $(x,\xi) \notin SS(\mathscr{F}_0)$. Then

$$\xi \in T_x^*M \cap SS(R\Gamma_Z(\mathscr{F}_0)) \subset (-\mathbb{R}_{\geq 0}\xi) + (T_x^*M \cap SS(\mathscr{F}_0)).$$

This implies $(x,\xi) \in SS(\mathscr{F}_0)$, which is a contradiction. This completes the proof. □

6.2. **The Structure of μsh_Λ.** Given $\Lambda \subset \dot{T}^*M$ a conical Lagrangian submanifold, we can associate the sheaf of categories μsh_Λ whose stalks are $\mathrm{Mod}(\Bbbk)$. The sheaf of categories is determined by (up to homotopy) the classifying map

$$\Lambda \to B\mathrm{Aut}_{\Bbbk}(\mathrm{Mod}(\Bbbk)) \xrightarrow{\sim} B\mathrm{Pic}(\Bbbk).$$

We would like to find out the obstruction of existence of global sections in μsh_Λ. The following theorem won't be proved.

Theorem 6.3 (Jin-Treumann; Jin)**.** *Let $\Bbbk$ be an E_2-spectrum and $\Lambda \subset \dot{T}^*M$ be a conical Lagrangian submanifold. Then the classifying map $\Lambda \to B\mathrm{Pic}(\Bbbk)$ factors as*

$$\Lambda \xrightarrow{G} U/O \xrightarrow{BJ} B\mathrm{Pic}(\mathbb{S}) \to B\mathrm{Pic}(\Bbbk),$$

where $G : \Lambda \to U/O$ is the stable Gaussian map and $BJ : U/O \to BPic(\mathbb{S})$ is the delooping of the J-homomorphism.

The obstruction classes are the Maslov class and the relative second Stiefel-Whitney class when $\Bbbk$ is an ordinary ring. We know that $U/O \simeq B(\mathbb{Z}\times BO)$. Since $\Bbbk$ is discrete, the J-homomorphism factors as

$$J : \mathbb{Z}\times BO \to \mathbb{Z}\times B(\mathbb{Z}/2\mathbb{Z}) \to \mathrm{Pic}(\Bbbk).$$

Now the map $L \to B(\mathbb{Z}\times B(\mathbb{Z}/2\mathbb{Z}))$ exactly defines the Maslov class and relative second Stiefel-Whitney class.

From now on we assume that $\Bbbk$ is a ring and show that in this setting the Maslov class and the relative second Stiefel-Whitney class are the obstructions.

6.2.1. *The Maslov Class and Maslov Sheaf.* Given $\Lambda \subset \dot{T}^*M$ a conical Lagrangian submanifold, let's consider the sheaf of categories μsh_Λ. In order to prove the existence of a global object in $D^b_\Lambda(M)$, first we try to consider a global object in $\mu sh_\Lambda(\Lambda)$. However there is an obstruction of the Maslov class.

Let

$$\sigma_M : LGr(T^*M) \to T^*M$$

be the Lagrangian Grassmannian of T^*M and

$$\sigma_M^0 : LGr^0(T^*M) \to T^*M$$

be the subbundle consisting of Lagrangian planes transverse to the Maslov cycle $T_p(T_x^*M)$.

Definition 6.2. *Let $\Lambda \subset \dot{T}^*M$ be a conical Lagrangian submanifold. Let*

$$\sigma_\Lambda : U_\Lambda \to \Lambda$$

be the subbundle of $LGr^0(T^*M)|_\Lambda$ consisting of Lagrangian planes transverse to Λ. Let

$$I_\Lambda = \{(l, \pi \circ \sigma_\Lambda(l)) | l \in U_\Lambda\} \subset U_\Lambda \times M,$$

$$J_\Lambda = \{(l, x; 0, \lambda\xi) | (x, \xi) = \sigma_\Lambda(l)\} \subset T^*(U_\Lambda \times M).$$

A C^∞ function φ on $U_\Lambda \times M$ is admissible if for any $(x, \xi) = \sigma_\Lambda(l)$,

$$\varphi_l(x) = 0,\ d\varphi_l(x) = \xi,\ (d\varphi_l)_x(T_xM) = l.$$

One can show that admissible functions exist. Since this is a purely differential topology exercise, we omit the proof here. For any sheaf $\mathscr{F}$, whenever we want to consider the microlocal behaviour at (x, ξ), we always consider a function φ so that $\varphi(x) = 0, d\varphi(x) = \xi$ and take the local cohomology

$$R\Gamma_{\varphi^{-1}([0,+\infty))}(\mathscr{F})_x.$$

Here we're just parametrizing such functions by Lagrangian subspaces so that everything can work out in the Lagrangian Grassmannian.

The following theorem is the preparation for constructing a map from $\mathscr{F} \in D^b_{(\Lambda)}(M)$ to a local system, which will be given by

$$\mathscr{F} \mapsto R\pi_{1,*}\mathscr{N}_{\varphi,\mathscr{F}} = R\pi_{1,*}(R\Gamma_{\varphi^{-1}([0,+\infty))}(\pi_2^{-1}\mathscr{F})_{I_\Lambda}).$$

(3) is saying that this is indeed a local system and (4) is saying that this local cohomology is indeed a parametrized version of the usual local cohomology we use.

Theorem 6.4. *Let φ be an admissible function and $\mathscr{F} \in D^b_{(\Lambda)}(M)$. Let $\pi_1 : U_\Lambda \times M \to U_\Lambda, \pi_2 : U_\Lambda \times M \to M$ be the projections. Then for*

$$\mathscr{M}_{\varphi,\mathscr{F}} = \mu hom(\Bbbk_{\varphi^{-1}([0,+\infty))}, \pi_2^{-1}\mathscr{F}) \in D^b(T^*(U_\Lambda \times M)),$$

$$\mathscr{N}_{\varphi,\mathscr{F}} = R\Gamma_{\varphi^{-1}([0,+\infty))}(\pi_2^{-1}\mathscr{F})_{I_\Lambda} \in D^b(U_\Lambda \times M),$$

there exists a neighbourhood V of I_Λ such that

(1). $\dot{T}^*V \cap \mathrm{supp}(\mathscr{M}_{\varphi,\mathscr{F}}) \subset J_\Lambda$ and $SS(\mathscr{M}_{\varphi,\mathscr{F}}|_{\dot{T}^*V}) \subset T^*_{J_\Lambda}T^*(U_\Lambda \times M)$;

(2). $R\dot{\pi}_*(\mathscr{M}_{\varphi,\mathscr{F}}|_{\dot{T}^*V}) \simeq \mathscr{N}_{\varphi,\mathscr{F}}$;

(3). $\dot{SS}(R\pi_{1,*}\mathscr{N}_{\varphi,\mathscr{F}}) = \emptyset$;

(4). $(R\pi_{1,*}\mathscr{N}_{\varphi,\mathscr{F}})_l \simeq R\Gamma_{\varphi_l^{-1}([0,+\infty))}(\mathscr{F})_{\tau_M(l)},\ \forall\, l \in U_\Lambda$.

Proof. (1). Note that there exists a neighbourhood V of $I_\Lambda \subset U_\Lambda \times M$ such that

(i). $SS^\infty(\Bbbk_{\varphi^{-1}([0,+\infty))}) \cap T^{*,\infty}V = \Lambda_\varphi^\infty$ is a submanifold;

(ii). $(U_\Lambda \times \Lambda) \cap \mathbb{R}_+\Lambda_\varphi = J_\Lambda$ and they intersect cleanly.

Then condition (1) will be satisfied. Indeed, as long as such a neighbourhood is constructed, we can use the estimate $SS(\mu hom(\Bbbk_{\varphi^{-1}([0,+\infty))}, \pi_2^{-1}\mathscr{F})) \subset C(SS(\Bbbk_{\varphi^{-1}([0,+\infty))}), SS(\pi_2^{-1}\mathscr{F})) \subset C(\mathbb{R}_+\Lambda_\varphi, U_\Lambda \times \Lambda)$.

(2). By Sato's exact triangle 3.14 we can obtain that

$$\Bbbk^\vee_{\varphi^{-1}([0,+\infty))} \otimes \pi_2^{-1}\mathscr{F} \otimes \Bbbk_{I_\Lambda} \to \mathscr{N}_{\varphi,\mathscr{F}} \to R\dot{\pi}_*(\mathscr{M}_{\varphi,\mathscr{F}})_{I_\Lambda} \xrightarrow{[1]}.$$

We know that $\varphi^{-1}([0,+\infty))$ is a smooth hypersurface, so $\Bbbk^\vee_{\varphi^{-1}([0,+\infty))} = \Bbbk_{\varphi^{-1}((0,+\infty))}$. However since $I_\Lambda \subset \varphi^{-1}(0)$, the first term in the exact triangle is zero.

(3). By part (1) $\mathscr{M}_{\varphi,\mathscr{F}}$ is locally constant along J_Λ. Since $J_\Lambda \to I_\Lambda$ has contractible fibers, by part (2) we obtain the result. Note that the projection $\pi_2 : I_\Lambda \to U_\Lambda$ is a diffeomorphism.

(4). We prove that $i_l : \{l\} \times M \to U_\Lambda \times M$ is non-characteristic for $R\Gamma_{\varphi^{-1}([0,+\infty))}(\pi_2^{-1}\mathscr{F})$ in a neighbourhood of x. By part (2) we estimate $R\dot{\pi}_*(\mu hom(\Bbbk_{\varphi^{-1}([0,+\infty))}, \pi_2^{-1}\mathscr{F}))$. Indeed we have

$$\pi_\pi \pi_d^{-1}(C(U_\Lambda \times \Lambda, \mathbb{R}_+\Lambda_\varphi)) \subset \pi_\pi \pi_d^{-1}(T^*_{J_\Lambda}(T^*(U_\Lambda \times \Lambda))) = T^*_{I_\Lambda}(U_\Lambda \times \Lambda).$$

Thus we can now conclude noncharacteristicity.

Non-characteristicity implies that we have

$$i_l^{-1}\mathscr{N}_{\varphi,\mathscr{F}} \simeq i_l^!\mathscr{N}_{\varphi,\mathscr{F}} \otimes \omega_{l/U_\Lambda}^{-1} = i_l^! R\Gamma_{\varphi^{-1}([0,+\infty))}(\pi_2^{-1}\mathscr{F}) \otimes \omega_{l/U_\Lambda}^{-1} \simeq R\Gamma_{\varphi_l^{-1}([0,+\infty))}(\mathscr{F})_x.$$

This completes the proof. □

Definition 6.3. *Let φ be an admissible function and $\pi_1 : U_\Lambda \times M \to U_\Lambda, \pi_2 : U_\Lambda \times M \to M$ be the projections. Then*

$$\begin{array}{rcl} m_\Lambda : D^b_{(\Lambda)}(M) & \to & D^b Loc(U_\Lambda) \quad ; \\ \mathscr{F} & \mapsto & R\pi_{1,*}(R\Gamma_{\varphi^{-1}([0,+\infty))}(\pi_2^{-1}\mathscr{F})_{I_\Lambda}) \, . \end{array}$$

For $l \in U_\Lambda$, the microlocal germ of $\mathscr{F}$ at l is $m_{\Lambda,l}(\mathscr{F}) = (m_\Lambda(\mathscr{F}))_l$.

Proposition 6.5. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian. Then*

$$m_{\Lambda_0} : D^b_{(\Lambda_0)}(M) \to D^b Loc(U_{\Lambda_0})$$

for all $\Lambda_0 \subset \Lambda$ open subsets induces a functor of stacks

$$m_\Lambda : \mu sh_\Lambda \to \sigma_{\Lambda,*}(D^b Loc_{U_\Lambda}).$$

In particular, for any $\mathscr{F}, \mathscr{G} \in D^b_{(\Lambda)}(M)$, there is a canonical isomorphism

$$\sigma_\Lambda^{-1} H^0 \mu hom(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} H^0 R\mathscr{H}om(m_\Lambda(\mathscr{F}), m_\Lambda(\mathscr{G})).$$

Proof. Let $\Lambda_0 \subset \Lambda$, and $\mathscr{F} \in D^b_{(\Lambda_0)}(M)$ such that $SS(\mathscr{F}) \cap \Lambda_0 = \emptyset$. Then $m_{\Lambda_0}(\mathscr{F}) = 0$. Hence the map m_{Λ_0} factors through $D^b(M;\Lambda_0)$. Therefore there is a natural functor between prestacks and after sheafification this gives the functor m_Λ. This is an isomorphism because of proposition 4.4. □

Although the previous proposition tells us that the functor m_Λ is fully faithful, it is not actually an equivalence. In fact we have to keep track of the module structure over the diagnol, which will be characterized by the Maslov sheaf. First recall that

$$D\mathscr{F} = R\mathscr{H}om(\mathscr{F}, \omega_M), \;\; D'\mathscr{F} = R\mathscr{H}om(\mathscr{F}, \Bbbk_M).$$

Proposition 6.6. *There exists a neighbourhood V of Δ_Λ in $\Lambda \times (-\Lambda)$ and a simple object $\mathscr{K}_{\Delta_\Lambda} \in \mu sh_V(V)$ so that for any $\Lambda_0 \subset \Lambda$, $\mathscr{F} \in D^b_{(\Lambda_0)}(M)$, there is a canonical morphism*

$$\mathscr{K}_{\Delta_\Lambda} \to (\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F})$$

which is an isomorphism when $\mathscr{F}$ is simple.

Proof. Choose a locally finite open cover $\{\Lambda_i\}_{i\in I}$ of Λ. Then it suffices to construct $\mathscr{K}_i$ on Λ_i and glue them using $\delta_{ij} : \mathscr{K}_i \xrightarrow{\sim} \mathscr{K}_j$ in $\mu sh_V(\Delta_{\Lambda_i \cap \Lambda_j})$.

Let $\mathscr{F}_i$ be a simple sheaf on Λ_i and let $\mathscr{K}_i = \pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F}$. Then we claim that there exists a (coherent choice of) $\delta_{ij} : \mathscr{K}_i \xrightarrow{\sim} \mathscr{K}_j$ in $\mu sh_V(\Lambda_i \cap \Lambda_j)$ that is compatible with

$\delta_{\mathscr{F}} \in H^0(\Delta_{\Lambda_i \cap \Lambda_j}, \mu hom(\Bbbk_\Delta, \pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F}))$ which is the image of identity under the map

$$\begin{aligned} Hom(\mathscr{F},\mathscr{F}) &\xrightarrow{\sim} \delta^! R\mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^!\mathscr{F})) \\ &\xrightarrow{\sim} Hom(\Bbbk_\Delta, R\mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^!\mathscr{F})) \\ &\to H^0(\Delta_{\Lambda_i\cap\Lambda_j}, \mu hom(\Bbbk_\Delta, \pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F})). \end{aligned}$$

By proposition 4.4, we know when $\mathscr{F}_i$ is simple, $\mu hom(\Bbbk_\Delta, \pi_1^{-1}\mathscr{F}_i \otimes \pi_2^{-1}D\mathscr{F}_i) \simeq \Bbbk_{\Delta_{\Lambda_i\cap\Lambda_j}}$. There is an isomorphism

$$\begin{aligned} &\mu hom(\pi_1^{-1}\mathscr{F}_i \otimes \pi_2^{-1}D\mathscr{F}_i, \pi_1^{-1}\mathscr{F}_j \otimes \pi_2^{-1}D\mathscr{F}_j)|_{\Delta_{\Lambda_i\cap\Lambda_j}} \\ &\quad\to \mu hom(\Bbbk_\Delta, \pi_1^{-1}\mathscr{F}_i \otimes \pi_2^{-1}D\mathscr{F}_i) \\ &\qquad \otimes \mu hom(\pi_1^{-1}\mathscr{F}_i \otimes \pi_2^{-1}D\mathscr{F}_i, \pi_1^{-1}\mathscr{F}_j \otimes \pi_2^{-1}D\mathscr{F}_j) \\ &\quad\to \mu hom(\Bbbk_\Delta, \pi_1^{-1}\mathscr{F}_j \otimes \pi_2^{-1}D\mathscr{F}_j). \end{aligned}$$

Now we just let δ_{ij} be the preimage of $\delta_{\mathscr{F}_j}$ under the isomorphism. □

Definition 6.4. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian, and $\tilde{U}_\Lambda = \Delta_\Lambda \times_{\Lambda\times(-\Lambda)} U_{\Lambda\times(-\Lambda)}$. Then the Maslov sheaf is*

$$\tilde{\mathscr{M}}_\Lambda = m_{\Lambda\times(-\Lambda)}(\mathscr{K}_{\Delta_\Lambda})|_{\tilde{U}_\Lambda} \in D^b Loc(\tilde{U}_\Lambda).$$

In addition, $\mathscr{M}_\Lambda = v_\Lambda^{-1}\tilde{\mathscr{M}}_\Lambda$ where v_Λ is the map $U_\Lambda \times_\Lambda U_\Lambda \to \tilde{U}_\Lambda$, $(l_1, l_2) \mapsto l_1 \oplus (-l_2)$.

Proposition 6.7. *Let $\tilde{U} \subset \tilde{U}_\Lambda$ be a connected component. Then $\mathscr{M}_\Lambda|_{\tilde{U}} \in DLoc(\tilde{U})$ is a rank 1 local system in degree $\tau_{\Lambda\times(-\Lambda)}(\tilde{U})/2$.*

Proof. Without loss of generality, we assume that $\Lambda = T_N^*M$. Consider locally a simple sheaf $\mathscr{F} \in D^b_{(\Lambda)}(M)$ such that

$$\mathscr{K}_{\Delta_\Lambda} \simeq \pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F}.$$

We may assume that $\mathscr{F} = \Bbbk_N$. Then since $\omega_{N|M} = \Bbbk[\dim N - \dim M]$ on N,

$$\mathscr{K}_{\Delta_\Lambda} \simeq \pi_1^{-1}\Bbbk_N \otimes \pi_2^{-1}D\Bbbk_N = \Bbbk_{N\times N}[\dim N].$$

Choose a function φ on $M \times M$ such that $\varphi|_{N\times N}$ is a non-degenerate quadratic form with signature $\tau_\Lambda(l)$. Then

$$m_\Lambda(L_{N\times N})_l \simeq R\Gamma_{\varphi^{-1}([0,+\infty))}(L_{N\times N})_{(x,x)} \simeq L[-\tau_\Lambda(l)/2 - \dim N].$$

Therefore since $L \simeq \Bbbk_{N\times N}[\dim N - \dim M]$ we know that $m_\Lambda(L_{N\times N})_l \simeq \Bbbk[-\tau_\Lambda(l)/2]$. □

Definition 6.5. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian. Then the sheaf of categories $\mu germ_\Lambda$ is defined by mapping all open subsets $\Lambda_0 \subset \Lambda$ to categories of pairs $(\mathscr{L}, u_{\mathscr{L}})$, where $\mathscr{L} \in DLoc_{U_\Lambda}(U_{\Lambda_0})$, and*

$$u_{\mathscr{L}} : \mathscr{M}_\Lambda \otimes \pi_2^{-1}\mathscr{L} \xrightarrow{\sim} \pi_1^{-1}\mathscr{L}$$

being commutative with the composition of Maslov sheaves.

Guillermou defined the Maslov sheaf as $\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D'\mathscr{F}$. However, I think instead the correct definition should be $\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F}$ in order to get the degree shifting $\tau_\Lambda(l)/2$. The point is that $D' \circ m_\Lambda = m_{-\Lambda} \circ D$ instead of $m_{-\Lambda} \circ D'$. Hence although what one really want is $\pi_1^{-1}\mathscr{L} \otimes \pi_2^{-1}D'\mathscr{L}$, when defining the Maslov sheaf one should really use $\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F}$.

We didn't define compositions between Maslov sheaves. Basically it is an isomorphism

$$\pi_{12}^{-1}\mathscr{M}_\Lambda \otimes \pi_{23}^{-1}\mathscr{M}_\Lambda \xrightarrow{\sim} \pi_{13}^{-1}\mathscr{M}_\Lambda,$$

where $\pi_{ij} : U_\Lambda^3 \to U_\Lambda^2$ is the projection to the i,j-th components. The crucial fact one will use is that

$$\begin{aligned}\pi_{12}^{-1}\mathscr{M}_\Lambda \otimes \pi_{23}^{-1}\mathscr{M}_\Lambda &\simeq \big((\pi_1^{-1}\mathscr{L}\otimes\pi_2^{-1}D'\mathscr{L})\otimes\Bbbk\big)\otimes\big(\Bbbk\otimes(\pi_1^{-1}\mathscr{L}\otimes\pi_2^{-1}D'\mathscr{L})\big)\\ &\simeq \pi_1^{-1}\mathscr{L}\otimes\pi_2^{-1}(\mathscr{L}\otimes D'\mathscr{L})\otimes\pi_3^{-1}D'\mathscr{L}\\ &\simeq \pi_1^{-1}\mathscr{L}\otimes\pi_3^{-1}D'\mathscr{L}\simeq\pi_{13}^{-1}\mathscr{M}_\Lambda.\end{aligned}$$

Theorem 6.8. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian. Then*

$$\begin{aligned} m_\Lambda :\ \mu sh_\Lambda(\Lambda_0) &\to \mu germ_\Lambda(\Lambda_0)\\ \mathscr{F} &\mapsto \Big(m_{\Lambda_0}(\mathscr{F}),\ u_{\mathscr{F}} : \mathscr{M}_\Lambda\otimes\pi_2^{-1}m_{\Lambda_0}(\mathscr{F}) \xrightarrow{\sim} \pi_1^{-1}m_{\Lambda_0}(\mathscr{F})\Big)\end{aligned}$$

is an equivalence of stacks.

Proof. First we check that this formula indeed defines a functor.

We consider an open cover with simple objects $\mathscr{F}_0$ in each $\Lambda_0 \subset \Lambda$. It can be easily seen that one can replace the functor in the statement by

$$\begin{aligned}\mu sh_\Lambda(\Lambda_0) &\to \mu germ_\Lambda(\Lambda_0)\\ \mathscr{F} &\mapsto \Big(m_{\Lambda_0}(\mathscr{F}_0)\otimes\sigma_{\Lambda_0}^{-1}\mu hom(\mathscr{F}_0,\mathscr{F}), u_{m_{\Lambda_0}(\mathscr{F}_0)}\otimes\mathrm{id}\Big).\end{aligned}$$

We know that as $\mathscr{F}_0$ is simple, $\mu hom(\mathscr{F}_0, -)$ is an equivalence.

Assume that Λ_0 is contractible and there is a section $s : \Lambda_0 \to U$. Let $\mathscr{L}_0 = m_{\Lambda_0}(\mathscr{F}_0)$. We prove that the functor

$$\begin{aligned} t_{\mathscr{L}_0} :\ D^bLoc_\Lambda(\Lambda_0) &\to \mu germ_\Lambda(\Lambda_0)\\ \mathscr{L} &\mapsto (\mathscr{L}_0\otimes\sigma_{\Lambda_0}^{-1}(\mathscr{L}), u_{\mathscr{L}_0}\otimes\mathrm{id}_{\mathscr{L}})\end{aligned}$$

is an equivalence. For $(\mathscr{L}, u_{\mathscr{L}}) \in \mu germ_\Lambda(\Lambda_0)$, define

$$j_{\mathscr{L}_0}(\mathscr{L}, u_{\mathscr{L}}) = s^{-1}(\mathscr{L}\otimes D'\mathscr{L}_0).$$

We prove that $t_{\mathscr{L}_0}$ and $j_{\mathscr{L}_0}$ are mutually inverse.

It suffices to show that there is an isomorphism $(\mathscr{L}, u_{\mathscr{L}}) \simeq t_{\mathscr{L}_0}\circ j_{\mathscr{L}_0}(\mathscr{L}, u_{\mathscr{L}})$. Since

$$u_{\mathscr{L}_0} : \mathscr{M}_\Lambda\otimes\pi_2^{-1}\mathscr{L}_0 \xrightarrow{\sim} \pi_1^{-1}\mathscr{L}_0,$$

we have an isomorphism $\mathscr{M}_\Lambda \simeq \pi_1^{-1}\mathscr{L}_0\otimes\pi_2^{-1}D'\mathscr{L}_0$. Hence (by dualizing $\pi_1^{-1}\mathscr{L}_0$) there exists

$$u_{\mathscr{L}}\otimes\mathrm{id}_{D'\mathscr{L}_0} : \pi_2^{-1}(\mathscr{L}\otimes D'\mathscr{L}_0) \xrightarrow{\sim} \pi_1^{-1}(\mathscr{L}\otimes D'\mathscr{L}_0).$$

Let $i : U_{\Lambda_0} \to U_{\Lambda_0}^2$ be $i(l) = (l, s\circ\sigma_{\Lambda_0}(l))$. By applying i^{-1} (and dualizing $D'\mathscr{L}_0$) we have

$$\mathscr{L} \xrightarrow{\sim} \mathscr{L}_0\otimes\sigma_{\Lambda_0}^{-1}s^{-1}(\mathscr{L}\otimes D'\mathscr{L}_0),$$

which essentially means that $\mathscr{L} \simeq \mathscr{L}_0\otimes\sigma_{\Lambda_0}^{-1}(j_{\mathscr{L}_0}(\mathscr{L}, u_{\mathscr{L}}))$. In addition, one can check that $u_{\mathscr{L}} = u_{\mathscr{L}_0}\otimes\mathrm{id}_{s^{-1}(\mathscr{L}\otimes D'\mathscr{L}_0)}$. This completes the proof. □

6.2.2. *Twisting of Local Systems.* Consider a local system $\mathscr{L} \in D^bLoc(M)$. Then it determines a homomorphism $\epsilon : \pi_1(M) \to GL(n,\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$. In the previous section, we've considered local systems on U_Λ. However, $\pi : U_\Lambda \to \Lambda$ may not have connected fibers. Hence we embed the fiber $U_{\Lambda,p}$ into a connected space and first consider the monodromy along the fibers $U_{\Lambda,p}$.

Our main goal will be to show that the monodromy information actually completely recovers the module structure over the diagnol or Maslov sheaf, so that we can throw away the notion of microlocal germs.

Definition 6.6. *Let $l \in U_{\Lambda,p}$ be a Lagrangian subspace transverse to both $T_p\Lambda$ and $T_pT^*_xM$. Let*

$$u_p(l) : T_pT^*_xM \to T_pT^*M \xrightarrow{\sim} T_p\Lambda \oplus l \to T_p\Lambda.$$

The embedding map is

$$\begin{aligned} i : U_{\Lambda,p} &\to \mathrm{Iso}_+(T_pT^*_xM \otimes \Lambda^n(T_pT^*_xM), T_p\Lambda \otimes \Lambda^n(T_p\Lambda)) \\ l \quad &\mapsto \qquad u_p(l) \otimes \Lambda^n u_p(l). \end{aligned}$$

For any connected component of $U_{\Lambda,p}$, let ϵ'_p be the composition

$$\pi_1(U_{\Lambda,p}) \to \pi_1(\mathrm{Iso}_+(T_pT^*_xM \otimes \Lambda^n(T_pT^*_xM), T_p\Lambda \otimes \Lambda^n(T_p\Lambda))) \xrightarrow{\epsilon_p} \mathbb{Z}/2\mathbb{Z},$$

where ϵ_p is the canonical morphism.

Proposition 6.9. *Let $\mathscr{F} \in D^b_{(\Lambda)}(M)$. For any connected component of $U_{\Lambda,p}$, the monodromy for $m_\Lambda(\mathscr{F})_{U_{\Lambda,p}}$ is ϵ'_p.*

Let $\Lambda = T^*_NM$ where $N = \{x | x_1 = ... = x_k = 0\}$. Then any Lagrangian subspace in U_Λ can be represented by a symmetric matrix A so that $A_{k+1,...,n|k+1,...,n} = (A_{ij})_{k+1 \le i,j \le n}$ is non-degenerate.

$$l_A = \{(\nu, A\nu) | \nu \in T_xM,\ A : T_xM \to T^*_xM\}.$$

Consider $u_p(A) : T_pT^*_xM \to T_p\Lambda$. If $u_p(A)(0, \eta) = (0, ..., x_{k+1}, ..., x_n, \xi_1, ..., \xi_k, 0, ..., 0)$, then

$$(0, \eta) = (\nu, A\nu) + (0, ..., x_{k+1}, ..., x_n, \xi_1, ..., \xi_k, 0, ..., 0).$$

As a result we have

$$\eta = (\xi_1, ..., \xi_k, 0, ..., 0) - A(0, ..., 0, x_{k+1}, ..., x_n).$$

If we consider the coordinate system $(\xi_1, ..., \xi_k, x_{k+1}, ..., x_n)$ for $T_p\Lambda$, then

$$u_p(A) = \begin{pmatrix} I_k & A_{1,...,k|k+1,...,n} \\ 0 & A_{k+1,...,n|k+1,...,n} \end{pmatrix}^{-1}.$$

This will enable us to do calculations in the following proof.

Proof. Without loss of generality, we assume that $\Lambda = T^*_NM$ and choose a local chart so that $N = \{x \in \mathbb{R}^n | x_1 = ... = x_k = 0\}$. Then $\mathscr{F} \simeq L_N$ near $p \in T^*_NM$. Since $\mathbb{Z}$ is the initial object in the category of rings, we assume that $\mathscr{F} \simeq \mathbb{Z}_N$ near $p \in T^*_NM$.

A Lagrangian plane in $U_{\Lambda,p}$ can be represented by a symmetric matrix A so that the determinant $\det((A_{ij})_{i,j \ge k+1}) \neq 0$. Fixing a connected component U is the same as fixing $\mathrm{sgn}(A)$. Let's say $\mathrm{sgn}(A) = 2l + k - n$. After choosing a base point

$$\mathrm{diag}(0, ..., 0, 1, ..., 1, -1, ..., -1).$$

$\pi_1(U)$ is generated by Γ_{ij} $(k + 1 \le i \le k + l, j \ge k + l + 1)$ where

$$(\Gamma_{ij}(\theta))_{pq} = \begin{cases} \pm\delta_{pq}, & (p, q) \neq (i, i), (i, j), (j, i) \text{ or } (j, j), \\ \sin\theta, & (p, q) = (i, j) \text{ or } (j, i), \\ \cos\theta, & (p, q) = (i, i), \\ -\cos\theta, & (p, q) = (j, j). \end{cases}$$

One can compute that $\epsilon'_p(\Gamma_{ij}) = -1$. However, the canonical map ϵ_p is surjective. Hence it suffices to show that the monodromy of $m_\Lambda(\mathbb{Z}_N)$ is -1.

Define the admissible function along Γ_{ij} to be

$$\varphi_\theta|_N = x^T_{k+1,...,n}\Gamma_{ij}(\theta)x_{k+1,...,n}.$$

Then by Morse theory, $\varphi_\theta^{-1}([0,+\infty)) \cap N$ is homotopy equivalent to the stable manifold of φ_θ. Now we know

$$m_\Lambda(\mathbb{Z}_N)_{\Gamma_{ij}(\theta)} \simeq R\Gamma_{\varphi_\theta^{-1}([0,+\infty))}(\mathbb{Z}_N) \simeq H_c^*(\varphi_\theta^{-1}([0,+\infty)); \mathbb{Z}).$$

The stable manifold of φ_θ is

$$V_\theta = \langle e_\theta, e_\alpha | k+1 \leq \alpha \leq k+l, \alpha \neq i \rangle, \; e_\theta = \cos(\theta/2)e_i + \sin(\theta/2)e_j.$$

Since the loop of unstable manifolds is not orientable, the monodromy is -1. □

We remark here that the space of non-degenerate symmetric matrices is a union of flag varieties. The fundamental group can thus be calculated.

Now for a fiber bundle $E \to M$, we define the category of local systems whose monodromy along fibers are always fixed to be $\epsilon : \underline{H_1}(E) \to \underline{\mathbb{Z}/2\mathbb{Z}}$.

Definition 6.7. *Let $E \to M$ be a fiber bundle and $\epsilon : \underline{H_1}(E) \to \underline{\mathbb{Z}/2\mathbb{Z}}$. Then the category $D^bLoc^\epsilon(E|M)$ consist of local systems $\mathscr{L}$ so that the monodromy of $\mathscr{L}|_{E_x}$ is ϵ_x. The sheaf of categories $DLoc^\epsilon_{E|M}$ is the sheafification of $U \mapsto D^bLoc^\epsilon(U)$.*

Lemma 6.10. *Let $E \to M$ be a fiber bundle and $U \to M$ a subbundle with connected fibers. Then there is an equivalence of stacks $D^bLoc^\epsilon_{E|M} \simeq D^bLoc^\epsilon_{U|M}$.*

Definition 6.8. *Let $\Lambda \subset \dot{T}^*M$ be a conical Lagrangian. Then*

$$\mathcal{I}_\Lambda = \mathrm{Iso}_+(i_\Lambda^*\pi^{-1}T^*M \otimes \Lambda^n(i_\Lambda^*\pi^{-1}T^*M), T\Lambda \otimes \Lambda^n(T\Lambda))$$

is a fiber bundle with fiber

$$\mathrm{Iso}_+(T_pT_x^*M \otimes \Lambda^n(T_pT_x^*M), T_p\Lambda \otimes \Lambda^n(T_p\Lambda)).$$

The tensor product here is added in order to make sure the resulting map is in $GL_+(n,\mathbb{R})$ since we don't know whether $u_p(l)$ itself is in $GL_+(n,\mathbb{R})$.

However, when passing from $DLoc^{\epsilon'}_{U_\Lambda|\Lambda}$ to $DLoc^{\epsilon'}_{\mathcal{I}_\Lambda|\Lambda}$, there is an issue, that is, U_Λ does not have connected fibers, so we won't get an equivalence of stacks. Hence instead we stabilize by $\Xi_N = T^*_{\mathbb{R}^{N-1}}\mathbb{R}^N \subset T^*\mathbb{R}^N$. Stabilization will introduce degree shifting of Maslov sheaves by the Maslov potential.

Lemma 6.11. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian, $\mu_\Lambda = 0$ and U_Λ has finitely many components $U_i\,(i \in I)$. Then there exists $N \in \mathbb{N}$ and components $V_i\,(i \in I)$ of U_{Ξ_N} such that $U_i \times V_i\,(i \in I)$ are in the same component $W \subset U_{\Lambda \times \Xi_N}$.*

Proof. Note that the Maslov index $\tau_{\Lambda\times\Xi_N} = \tau_\Lambda - \tau_{\Xi_N}$ classifies all connected components. The Cech cocycle $(\tau_\Lambda(U_i) - \tau_\Lambda(U_j))_{i,j\in I}$ in $H^1(\Lambda;\mathbb{Z})$ is twice the Maslov class, which is equal to zero. Hence one can find $(n_i)_{i\in I}$ such that

$$\tau_\Lambda(U_i) - \tau_\Lambda(U_j) = 2(n_i - n_j).$$

Now it suffices to choose N large enough so that the Maslov index on Ξ_N can be large enough, and consider V_i to be the component with Maslov index $2n_i$. □

We explain here why $(\tau_\Lambda(U_i) - \tau_\Lambda(U_j))_{i,j\in I}$ defines the Maslov class in $H^1(\Lambda;\mathbb{Z})$ (the reader may refer to *Geometric Asymptotics*, Guillemin & Sternberg, Chapter IV, Section 3 for a complete proof). Firstly, each component $\Lambda_i = \pi(U_i)$ admits a grading. This can be seen through applying the map

$$u_i : i_\Lambda^*T^*M \to T\Lambda, \; u_p(U_i) : T_pT_x^*M \to T_p\Lambda.$$

If these gradings or trivializations can be glued together depends on the transition functions $(\mu_{ij})_{i,j\in I} = (u_i \circ u_j^{-1})_{i,j\in I}$. Note that the Maslov class is the pull-back of the generator $\sigma \in H^1(S^1)$ by

$$\theta_\Lambda : \Lambda \to LGr(T^*M) \to S^1,\ U_p \mapsto \det(U_p)^2$$

where $U_p \in U(n)/O(n) \simeq LGr_p(T^*M)$. Consider the commutative diagram

$$\begin{array}{ccc} H^0(\Lambda; C_\Lambda^\times) & \longleftarrow & H^0(S^1; C_{S^1}^\times) \\ \downarrow & & \downarrow \\ H^1(\Lambda; \mathbb{Z}) & \longleftarrow & H^1(S^1; \mathbb{Z}) \end{array}$$

where $C_X^\times$ is the sheaf of S^1-valued functions on X, and the vertical arrows are given by $z \mapsto (\log z)/2\pi\sqrt{-1}$. Since the generator in $H^1(S^1;\mathbb{Z})$ is given by the identity section in $H^0(S^1; C_{S^1}^\times)$, we know that the image of the section $\theta_\Lambda \in H^0(\Lambda; C_\Lambda^\times)$ in $H^1(\Lambda;\mathbb{Z})$ is just the Maslov class. However, we claim that

$$\frac{1}{2\pi\sqrt{-1}} \log\left(\det(\mu_{ij})^2/|\det(\mu_{ij})|^2\right) = \frac{1}{2}(\tau_\Lambda(U_i) - \tau_\Lambda(U_j)).$$

Indeed, the left hand side is equal to the number of eigenvalues that change from negative to positive, and so is the right hand side. This shows that $((\tau_\Lambda(U_i) - \tau_\Lambda(U_j))/2)_{i,j\in I}$ is indeed the Maslov class.

Lemma 6.12. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian, $\mu_\Lambda = 0$ and U_Λ has finitely many components. Suppose $\mathscr{F} \in D^b_{(\Lambda)}(M)$ is simple along Λ. Then there exists $\mathscr{L} \in Loc^\epsilon(\mathcal{I}_\Lambda|\Lambda)$ such that*

(1). $m_\Lambda(\mathscr{F})|_U \simeq \mathscr{L}|_U[d_U]$ for some $d_U \in \mathbb{Z}$ where $U \subset U_\Lambda$ is a connected component and $d_U - d_{U'} = (\tau_\Lambda(U) - \tau_\Lambda(U'))/2$;

(2). $m_{\Lambda\times(-\Lambda)}(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}D\mathscr{F})|_{\tilde{U}_\Lambda^{2l}} \simeq (\pi_1^{-1}\mathscr{L} \otimes \pi_2^{-1}D'\mathscr{L})|_{\tilde{U}_\Lambda^{2l}}[-l]$ where $\tilde{U}_\Lambda^{2l}$ is the component with Maslov potential $2l$.

Proof. We only prove (1) (the proof for (2) is similar). Let W be as in the previous lemma. Let $\mathscr{L} = m_{\Lambda\times\Xi_N}(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_N)|_W[d]$ be a local system concentrated in degree 0. Since $D^bLoc^\epsilon(\mathcal{I}_{\Lambda\times\Xi_N}|\Lambda\times\Xi_N) \simeq D^bLoc^\epsilon(W|\Lambda\times\Xi_N)$, $\mathscr{L}$ extends to $\mathcal{I}_{\Lambda\times\Xi_N}$. However, one can prove that there is an unique isomorphism

$$m_{\Xi_N}(\Bbbk_N)|_{V^k_{\Xi_N}} \simeq \mathscr{L}_N|_{V^k_{\Xi_N}}[\lfloor k/2 \rfloor].$$

Note that $\mathscr{L}_N$ is the only rank 1 object in $Loc^\epsilon(\Xi_N)$. Therefore we have a unique decomposition $\mathscr{L} \simeq \pi_1^{-1}\mathscr{L}_\Lambda \otimes \pi_2^{-1}\mathscr{L}_N$. Therefore the isomorphism $m_{\Lambda\times\Xi_N}(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_N)|_{U_i\times V^k_{\Xi_N}}[d]$ and $m_{\Xi_N}(\Bbbk_N)|_{V^k_{\Xi_N}} \simeq \mathscr{L}_N|_{V^k_{\Xi_N}}[\lfloor k/2\rfloor]$ together gives the isomorphism we want. □

Now we can take into account the degree shifting and modify our definition of microlocal germs. Similarly one can define the composition of Maslov sheaves, which will again be omitted though.

Definition 6.9. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian. Then the modified Maslov sheaf is $\mathscr{M}'_\Lambda|_U = \mathscr{M}_\Lambda[\tau_{\Lambda\times(-\Lambda)}(U)/2]$ where $U \subset U^2_\Lambda$ is a connected component. The sheaf of categories $\mu germ'_\Lambda$ is defined accordingly.*

Now one can check that the twisting by the monodromy along fibers actually recovers the module structure over the diagnol or the Maslov sheaf, in other words, the information of microlocal germs.

Theorem 6.13. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian. Then $D^bLoc^\epsilon_{\mathcal{I}_\Lambda|\Lambda} \to \mu germ'_\Lambda$ is an equivalence of stacks.*

Proof. It suffices to recover the isomorphism

$$u'_{\mathscr{L}} : \mathscr{M}'_\Lambda \otimes \pi_2^{-1}\mathscr{L} \to \pi_1^{-1}\mathscr{L}.$$

By passing to a small open subset, we may assume that Λ is contractible. Then $DLoc^\epsilon(\mathcal{I}_\Lambda|\Lambda)$ has a unique object $\mathscr{L}_0$ with stalk $\mathbb{Z}$. One can write $\mathscr{L} = \mathscr{L}_0 \otimes \sigma_\Lambda^{-1}\mathscr{L}'$ and now the result becomes trivial. ☐

6.2.3. *Maslov Class and Stiefel-Whitney class.* In the previous section we are assuming that the Maslov class $\mu_\Lambda = 0$. Here we consider the Maslov class and show that the obstruction of existence of global sections is exactly characterized by the Maslov class (determining the degree shifting) and the relative second Stiefel-Whitney class (determining the twisting).

Definition 6.10. *Let $\mathcal{C}$ be the sheaf of derived (differential graded) categories, $\{\Lambda_i\}_{i\in I}$ an open cover of Λ, and $a = (a_{ij})_{i,j\in I}$ a degree 1 Cech cocycle in $\underline{\mathbb{Z}}$. Then $\mathcal{C}_a$ is the sheaf of categories glued by $\mathcal{C}_i$ and $\mathcal{C}_i|_{\Lambda_i\cap\Lambda_j} \to \mathcal{C}_j|_{\Lambda_i\cap\Lambda_j}$, $\mathscr{F} \mapsto \mathscr{F}[a_{ij}]$.*

Lemma 6.14. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian with Maslov class μ_Λ. Then there is an open cover $\{\Lambda_i\}_{i\in I}$ of Λ and a representative $\mu = (\mu_{ij})_{i,j\in I}$ of μ_Λ such that $\mu germ_\Lambda \simeq (\mu germ'_\Lambda)_\mu$.*

Proof. Consider an open cover $\{\Lambda_i\}_{i\in I}$ coming from connected components $\{U_i\}_{i\in I}$ of U_Λ with simple sheaves $\mathscr{F}_i$ along Λ_i. Then $m_\Lambda(\mathscr{F}_i)$ will be concentrated in degree d_{U_i}. Then $(d_{U_i} - d_{U_j})_{i,j\in I}$ represents the Maslov class of Λ. The gradings work out well because

$$d_{U_i} - d_{U_j} = \frac{1}{2}(\tau_\Lambda(U_i) - \tau_\Lambda(U_j)) = \frac{1}{2}\tau_{\Lambda\times(-\Lambda)}(U_i \times (-U_j)).$$

Now we can define the equivalence $\mu germ_\Lambda(\Lambda_i) \to \mu germ'_\Lambda(\Lambda_i)$ locally by $\mathscr{L}|_{U_i} \mapsto \mathscr{L}[-d_{U_i}]|_{U_i}$. This can be glued to a global isomorphism. ☐

Theorem 6.15. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian with Maslov class μ_Λ. Then there is an open cover $\{\Lambda_i\}_{i\in I}$ of Λ and a representative $\mu = (\mu_{ij})_{i,j\in I}$ of μ_Λ such that*

$$\mu sh_\Lambda \simeq (D^bLoc^\epsilon_{\mathcal{I}_\Lambda|\Lambda})_\mu.$$

In other words, there exists a global section in μsh_Λ iff $\mu_\Lambda = 0$.

Definition 6.11. *Let $E_{1,2} \to \Lambda$ be vector bundles of the same rank and $\{\Lambda_i\}_{i\in I}$ be a good cover of Λ. Let $u = (u_{ij})_{i,j\in I}$ be a Cech cochain given by isomorphisms $u_{ij} : \mathscr{L}_i|_{\Lambda_{ij}} \xrightarrow{\sim} \mathscr{L}_j|_{\Lambda_{ij}}$ of rank 1 local systems twisted by*

$$\epsilon : \underline{H}_1(\mathcal{I}_{E_1,E_2,\Lambda}) = \underline{H}_1\left(\mathrm{Iso}_+(E_1 \otimes \Lambda^n(E_1), E_2 \otimes \Lambda^n(E_2))\right) \to \underline{\Bbbk}^\times,$$

and let $w = (w_{ijk})_{i,j,k\in I} = (u_{ki} \circ u_{jk} \circ u_{ij})_{i,j,k\in I}$. Then the relative second Stiefel-Whitney class of E_1 and E_2, denoted by $rw_2(E_1, E_2)$, is the class represented by w in $H^2(\Lambda; \mathbb{Z}/2\mathbb{Z})$.

We claim that when $\Bbbk = \mathbb{Z}$ ($\Bbbk^\times = \mathbb{Z}/2\mathbb{Z}$), $rw_2(E_1, E_2) = w_2(E_2 \otimes \Lambda^n E_2) - w_2(E_1 \otimes \Lambda^n E_1)$. First when E_1 is trivial, $\mathcal{I}_{\Bbbk^n_\Lambda, E_2, \Lambda}$ is the principal bundle whose associate bundle is $E_2 \otimes \Lambda^n E_2$. $w_2(E_2 \otimes \Lambda^n E_2)$ is the obstruction for the principal $SO(n)$-bundle to be lifted to a principal $Spin(n)$-bundle, so let $v_{ij} : \Lambda_{ij} \to SO(n)$ be the transition function, and $\tilde{v}_{ij} : \Lambda_{ij} \to Spin(n)$ be the lifting, then

$$\tilde{w} = (\tilde{w}_{ijk})_{i,j,k\in I} = (\tilde{v}_{ki} \circ \tilde{v}_{jk} \circ \tilde{v}_{ij})_{i,j,k\in I}$$

defines the second Stiefel-Whitney class. On the other hand, note that for an ϵ-twisted line bundle on $\Lambda_{ij} \times SO(n)$, its corresponding principal bundle is, since ϵ is nontrivial, $\Lambda_{ij} \times Spin(n)$. Hence

$$u_{ij} : \Lambda_{ij} \times Spin(n) \to \Lambda_{ij} \times Spin(n)$$

is indeed the lifting of the transition function on the $SO(n)$-principal bundle. Therefore it defines the second Stiefel-Whitney class.

Theorem 6.16. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conical Lagrangian. Then there exists a simple global section in μsh_Λ iff $rw_2(i_\Lambda^*\pi^{-1}T(T^*M), T\Lambda) = 0$.*

6.3. **Convolution and Anti-microlocalization.** From this section on, we assume that the manifold we're considering is $M \times \mathbb{R}$, where the $\mathbb{R}$ factor is required as the direction of the Reeb vector field, and thus the conical Lagrangian lives in $\dot{T}^*_{\tau>0}(M \times \mathbb{R})$ (where τ is the fiber coordinate for the $\mathbb{R}$ component).

By Theorem 6.8, we know that there is a global simple object $\overline{\mathscr{F}} \in \mu sh_\Lambda(\Lambda)$. There exists an open covering $\{\Lambda_i\}_{i\in I}$ of Λ such that $\overline{\mathscr{F}}|_{\Lambda_i}$ is represented by $\mathscr{F}_i \in D^b_{(\Lambda_i)}(M)$, and the transition functions are

$$\bar{u}_{ij} \in H^0(\Lambda_i \cap \Lambda_j, \mu hom(\mathscr{F}_i, \mathscr{F}_j)|_{\Lambda_i \cap \Lambda_j}).$$

Remember our goal is to give a sheaf $\mathscr{F} \in D^b_\Lambda(M)$. Therefore we need to represent $\bar{u}_{ij}$ by elements in $Hom_{D^b_{(\Lambda_i \cap \Lambda_j)}(M)}(\mathscr{F}_i, \mathscr{F}_j)$ in order to glue a global object.

In fact, we would like to make use of the $\mathbb{R}$ direction, or the Reeb direction and consider translation along that direction. In addition we want to consider all translations at a time, which will require another extra $\mathbb{R}_+$ factor to encode how much we are translating. We will construct a functor Ψ_U such that

$$\varinjlim_{\epsilon>0} Hom_{D^b(U\times(0,\epsilon))}(\Psi_U^\epsilon(\mathscr{F}), \Psi_U^\epsilon(\mathscr{G})) \simeq H^0(\dot{T}^*U, \mu hom(\mathscr{F}, \mathscr{G})).$$

Such a functor Ψ_U will be built by convolution. The key idea is coming from Tarmarkin, where he showed that the convolution functor

$$\Bbbk_{[0,+\infty)} \star - : D^b(M \times \mathbb{R}) \to D^b(M \times \mathbb{R}); \;\; \mathscr{F} \mapsto \Bbbk_{[0,+\infty)} \star \mathscr{F}$$

is the projection from $D^b(M \times \mathbb{R})$ to the left orthogonal complement of $D^b_{\tau\leq 0}(M \times \mathbb{R})$, i.e. $D^b(M; T^*_{\tau\leq 0}(M \times \mathbb{R}))$ is realized as the left orthogonal complement of $D^b_{\tau\leq 0}(M)$.

Definition 6.12. *Let $U \subset M \times \mathbb{R}$ be an open subset. Define the projections*

$$q : M \times \mathbb{R} \times \mathbb{R}_+ \to M \times \mathbb{R}, \; (x, t, u) \mapsto (x, t)$$
$$r : M \times \mathbb{R} \times \mathbb{R}_+ \to M \times \mathbb{R}, \; (x, t, u) \mapsto (x, t - u).$$

Then let

$$U_+ = r^{-1}(U) \cap q^{-1}(U) = \{(x, t, u) \in M \times \mathbb{R} \times \mathbb{R}_+ | x \times [t - u, t] \subset U\}.$$

and write $q_U = q|_{U_+}, r_U = r|_{U_+}$. Let $\gamma = \{(t, u) \in \mathbb{R} \times \mathbb{R}_+ | 0 \leq t < u\}$, $\gamma_1 = \{(t, u) \in \mathbb{R} \times \mathbb{R}_+ | t = u\}$. Let $s : M \times \mathbb{R}^2 \times \mathbb{R}_+ \to M \times \mathbb{R} \times \mathbb{R}_+$ be the addition. Then

$$\Psi_U(\mathscr{F}) = \Bbbk_\gamma \star' \mathscr{F} = Rs_!(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_\gamma)_{U_+}.$$

One may wonder why we restrict to U_+ after applying the convolution functor. This can be illustrated from the following diagram, which shows what will happen when we finally start to glue.

In fact, when gluing together $U_1 \times \mathbb{R}$ and $U_2 \times \mathbb{R}$, the singular supports in different pieces may overlap because extra Reeb chords may be created after gluing. Restricting to U_{1+}, U_{2+} is preventing any extra Reeb chord from shrinking (However, this does not mean that in a single piece U_1 or U_2, there cannot exist any Reeb chords that shrink to a point).

As a counterexample, one may consider a stablized Legendrian knot in $T^{*,\infty}\mathbb{R}^2$. Consider the doubling link constructed by pushing forward along the Reeb direction by $\epsilon > 0$. When ϵ is small, there is a sheaf with singular support in the doubled link. This can be down by

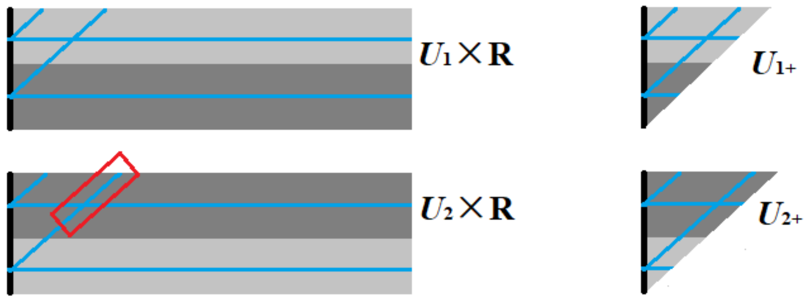

FIGURE 7. Let U_1, U_2 be two intervals and Λ consists of some isolated points. After applying the convolution functor, the singular support of the sheaf will be the blue rays. In the dark grey region where $U_1 \times \mathbb{R}$ and $U_2 \times \mathbb{R}$ overlap, one can see that the singular support don't glue. Hence one should restrict to U_{1+}, U_{2+} where such things won't happen.

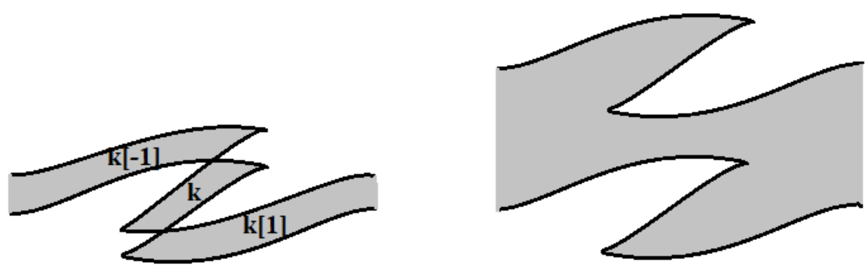

FIGURE 8. On the left is the front projection of a two copy of a stablized knot when ϵ is small, together with a sheaf whose singular support is the two copy. On the right is the front projection of a two copy when ϵ is large, where no such sheaf exists.

gluing two simple sheaves on two different charts of the zig-zag (since there is no simple sheaves along a zig-zag, one has to split into two different charts). However when ϵ is large enough so that the front projections of the two copies become disjoint, then it is well known that no such sheaves can exist (see, for example, Shende-Treumann-Zaslow). And this is exactly because when $\epsilon > 0$ is large, **Reeb chords between the two different charts** will shrink and create double points in the process.

However, one may wonder why this is not the case for some other Legendrian knots, for example the standard unknot. There the unique Reeb chord may also shrink, however, when $\epsilon > 0$ is large one can still find a sheaf whose singular support is the two copy of unknot. This is because one can choose an open cover consisting of a unique open subset U and a sheaf that represents the simple object in μsh. Then one does not need to glue. Here one can see that in a single chart Reeb chords are allowed to shrink, as long as no extra Reeb chords connecting two different charts shrink.

Lemma 6.17. *Let $a < b$ and $\mathscr{F} \in D^b_{\tau \geq 0}(\mathbb{R})$. Then $\Bbbk_{(a,b]} \star' \mathscr{F} = 0$.*

Proof. Fix $x \in \mathbb{R}$. By definition we have

$$\begin{aligned}(\Bbbk_{(a,b]} \star' \mathscr{F})_x &= R\Gamma_c(s^{-1}(x), \pi_1^{-1}\Bbbk_{(a,b]} \otimes \pi_2^{-1}\mathscr{F}|_{s^{-1}(x)}) \\ &= R\Gamma_c(\mathbb{R}, \mathscr{F} \otimes \Bbbk_{[x-b,x-a)}).\end{aligned}$$

Notice that there is an exact triangle

$$R\Gamma([x-b, x-a], \mathscr{F}) \to \mathscr{F}_{x-a} \to R\Gamma_c(\mathbb{R}, \mathscr{F} \otimes \Bbbk_{[x-b,x-a)}) \xrightarrow{[1]}$$

and the first map is an isomorphism by microlocal cutoff lemma. This proves the result. □

Lemma 6.18. *For $\mathscr{F} \in D^b_{\tau\geq 0}(M \times \mathbb{R})$, the canonical morphism $r_U^{-1}\mathscr{F}[-1] \to \Bbbk_{\gamma^\circ} \star' \mathscr{F}$ is an isomorphism, and there is an distinguished triangle*

$$r_U^{-1}\mathscr{F}[-1] \to \Psi_U(\mathscr{F}) \to q_U^{-1}\mathscr{F} \xrightarrow{[1]}.$$

Proof. Note that $r_U^{-1}\mathscr{F} \simeq \Bbbk_{\gamma_1} \star' \mathscr{F}$. Let $\gamma' = \bar{\gamma}\backslash\gamma_0$. We have a distinguished triangle

$$r_U^{-1}\mathscr{F}[-1] \to \Bbbk_{\gamma^\circ} \star' \mathscr{F} \to \Bbbk_{\gamma'} \star' \mathscr{F} \xrightarrow{[1]}.$$

To prove the first morphism is an isomorphism, it suffices to show that $\Bbbk_{\gamma'} \star' \mathscr{F} \simeq 0$. This is because

$$(\Bbbk_{\gamma'} \star' \mathscr{F})_{(x,u)} \simeq \Bbbk_{(0,u]} \star' \mathscr{F} \simeq 0.$$

Now the theorem follows from the standard exact triangle

$$\Bbbk_{\gamma^\circ} \star' \mathscr{F} \to \Bbbk_{\gamma} \star' \mathscr{F} \to \Bbbk_{\gamma_0} \star' \mathscr{F} \xrightarrow{[1]}$$

and the fact that $q_U^{-1}\mathscr{F} \simeq \Bbbk_{\gamma_0} \star' \mathscr{F}$. □

Lemma 6.19. *Let $\mathscr{F} \in D^b(U)$ and $V \subset U$ be an open subset. Assume that for any $x \in M$, $V_x = V \cap (\{x\} \times \mathbb{R})$, $\mathscr{F}|_{V_x}$ is locally constant. Then $\Psi_U(\mathscr{F})|_V \simeq 0$. In particular if $SS(\mathscr{F}|_V) \subset V \subset T^*V$, then $\Psi_U(\mathscr{F})|_V \simeq 0$.*

Proof. Note that $\Psi_{V_x}(\mathscr{F}|_{V_x}) \simeq \Psi_U(\mathscr{F})|_{V_x}$ and V_x is a disjoint union of open intervals. Hence the result follows from direct computation. □

Remember that our goal is to build a bridge between the $RHom(-,-)$ of convolutions and $\mu hom(-,-)$. Define

$$i : U \to U \times \mathbb{R}_{\geq 0},\ (x,t) \mapsto (x,t,0); \quad j : U \times \mathbb{R}_+ \to U \times \mathbb{R}_{\geq 0},\ (x,t,u) \mapsto (s,t,u).$$

Our goal is to show the following isomorphism:

Theorem 6.20. *Let $U \subset M \times \mathbb{R}$ be open, $\mathscr{F} \in D^b_{\tau\geq 0}(U)$ and $\mathscr{G} \in D^b_{\tau>0}(U)$. Then we have a natural isomorphism*

$$i^{-1}Rj_*R\mathscr{H}om(\Psi_U(\mathscr{F}), \Psi_U(\mathscr{G})) \simeq R\dot{\pi}_{U,*}\mu hom(\mathscr{F}, \mathscr{G}).$$

In particular for $Z \subset U$ being compact we have

$$H^k(\dot{\pi}_U^{-1}(Z), \mu hom(\mathscr{F}, \mathscr{G})) \simeq \varinjlim_{W:\, Z\subset\subset W\subset\subset U\times\mathbb{R}_{\geq 0}} Hom(\Psi_U(\mathscr{F})|_W, \Psi_U(\mathscr{G})[k]|_W).$$

Therefore, Ψ_U defines a fully faithful functor between the sheaves of categories

$$\Psi : \mu sh_\Lambda(\Lambda) \to \varinjlim_{\epsilon>0} sh_{q_d q_\pi^{-1}(\Lambda)\cup r_d r_\pi^{-1}(\Lambda)}(M \times \mathbb{R} \times (0,\epsilon)).$$

We use the exact triangle in Lemma 6.17 and hence we prove two Propositions 6.21 and 6.24 separately. Here is the first proposition we need.

Proposition 6.21. *Let $\mathscr{F}, \mathscr{G} \in D^b(U)$. Then there is a natural isomorphism*

$$i^{-1}Rj_*R\mathscr{H}om(\Psi_U(\mathscr{F}), q_U^{-1}\mathscr{G}) \xrightarrow{\sim} R\dot{\pi}_{U,*}\mu hom(\mathscr{F}, \mathscr{G}).$$

We need a few lemmas.

Lemma 6.22. *Let $\mathscr{F} \in D^b_{\tau\geq 0}(U), \mathscr{G} \in D^b_{\tau>0}(U)$. Then there are natural isomorphisms*

$$\mu hom(\Psi_U(\mathscr{F}), q_U^{-1}\mathscr{G})|_{\dot{T}^*U_+} \xrightarrow{\sim} \mu hom(q_U^{-1}\mathscr{F}, q_U^{-1}\mathscr{G})|_{\dot{T}^*U_+} \xrightarrow{\sim} Rq_{U,d,!}q_{U,\pi}^{-1}\mu hom(\mathscr{F}, \mathscr{G})|_{\dot{T}^*U_+}.$$

Proof. For the first isomorphism, using the exact triangle in Lemma 6.17, it suffices to show that

$$\mu hom(r_U^{-1}\mathscr{F}[-1], q_U^{-1}\mathscr{G})|_{\dot{T}^*U_+} \simeq 0.$$

This follows from the fact that $\mathrm{supp}(\mu hom(r_U^{-1}\mathscr{F}[-1], q_U^{-1}\mathscr{G})) \subset SS(r_U^{-1}\mathscr{F}) \cap SS(q_U^{-1}\mathscr{G})$. The second isomorphism follows from 3.13. □

Lemma 6.23. *Let $\mathscr{F}, \mathscr{G} \in D^b(U)$. Then there are natural isomorphisms*

$$i^{-1}Rj_*R\mathscr{H}om(\Psi_U(\mathscr{F}), q_U^{-1}\mathscr{G}) \xrightarrow{\sim} i^{-1}Rj_*R\dot{\pi}_{U_+,*}(\mu hom(\Psi_U(\mathscr{F}), q_U^{-1}\mathscr{G})|_{\dot{T}^*U_+}).$$

Proof. Write $R\mathscr{H}om'(\mathscr{F}, \mathscr{G}) = \delta^{-1}R\mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^{-1}\mathscr{G})$. By Sato's exact triangle 3.14, it suffices to show that

$$i^{-1}Rj_*(R\mathscr{H}om'(\Psi_U(\mathscr{F}), q_U^{-1}\mathscr{G})) \simeq 0.$$

Viewing γ as a subset in $M \times \mathbb{R}^2$, we have

$$i^{-1}Rj_*(R\mathscr{H}om'(\Psi_U(\mathscr{F}), q_U^{-1}\mathscr{G})) \simeq i^{-1}R\Gamma_{M\times\mathbb{R}\times\mathbb{R}_+}(R\mathscr{H}om'(\Psi'_U(\mathscr{F}), q_U^{-1}\mathscr{G}))$$

where $\Psi'_U(\mathscr{F}) = Rs_!(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_\gamma)$.

Note that $R\Gamma_{M\times\mathbb{R}\times\mathbb{R}_+}\mathscr{H} = \mathscr{H}_{M\times\mathbb{R}\times\mathbb{R}_{\geq 0}}$ when $SS(\mathscr{H}) \subset \{(x,\xi;t,\tau) | \tau \geq 0\}$. Hence here we estimate $SS(R\mathscr{H}om'(\Psi'_U(\mathscr{F}), q_U^{-1}\mathscr{G}))$. Using the fact that $SS(\Bbbk_\gamma) \subset \{(s,t,\sigma,\tau) | -\tau \leq \sigma \leq 0\}$, we can estimate

$$SS(\Psi'_U(\mathscr{F})|_V) \subset \{(x,\xi;s,t,\sigma,\tau) | -\tau \leq \sigma \leq 0\}.$$

On the other hand, $SS(q_U^{-1}\mathscr{G}) \subset \{(x,\xi;t,\tau) | \tau = 0\}$. Therefore

$$SS(\mathscr{H}) \subset \{(x,\xi;s,t,\sigma,\tau) | \sigma \geq 0\}.$$

Now the right hand side is just $R\mathscr{H}om'(i^{-1}\Psi'_U(\mathscr{F}), i^{-1}q_U^{-1}\mathscr{G})$. Let $i' : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2 \times \mathbb{R}_+, (x,s,t) \mapsto (x,s,t,0)$ and $s' : M \times \mathbb{R}^2 \to M \times \mathbb{R}, (x,s,t) \mapsto (s, s+t)$ (such that $s \circ i' = i \circ s'$). Hence by the base change formula

$$i^{-1}\Psi'_U(\mathscr{F}) = i^{-1}Rs_!(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_\gamma) \simeq Rs'_!(i')^{-1}(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_\gamma) \simeq 0$$

because $\Bbbk_\gamma|_{\mathbb{R}\times\{0\}} = 0$. This completes the proof. □

Proof of Proposition 6.21. By the previous lemmas, it suffices to show that

$$i^{-1}Rj_*R\pi_{U_+,*}(Rq_{U,d,!}q_{U,\pi}^{-1}\mu hom(\mathscr{F}, \mathscr{G})|_{\dot{T}^*U_+}) \simeq R\dot{\pi}_{U,*}\mu hom(\mathscr{F}, \mathscr{G}).$$

This is because by base change formula, since $\dot{\pi}_U \circ q_{U,\pi} = q_U \circ \dot{\pi}_{U_+} \circ q_{U,d}$, we have

$$\begin{aligned} R\pi_{U_+,*}(Rq_{U,d,!}q_{U,\pi}^{-1}\mathscr{H}|_{\dot{T}^*U_+}) &\simeq R(\pi_{U_+} \circ q_{U,d})_! q_{U,\pi}^{-1}\mathscr{H}|_{\dot{T}^*U} \\ &\simeq q_U^{-1}R\dot{\pi}_{U,!}\mathscr{H}|_{\dot{T}^*U} \simeq q_U^{-1}R\dot{\pi}_{U,*}\mathscr{H}|_{\dot{T}^*U}. \end{aligned}$$

Now it suffices to show that

$$i^{-1}Rj_*q_U^{-1}\mathscr{K} \simeq \mathscr{K}.$$

Note that $i^{-1}Rj_*q_U^{-1} \simeq i^{-1}R\Gamma_{U\times\mathbb{R}_+} \circ \pi_1^{-1}$. Since $SS(\pi_1^{-1}\mathscr{F}) \subset T^*U \times \mathbb{R}$, we have

$$i^{-1}R\Gamma_{U\times\mathbb{R}_+}(\pi_1^{-1}\mathscr{F}) \simeq i^{-1}(\pi_1^{-1}\mathscr{F}|_{U\times\mathbb{R}_{\geq 0}}) \simeq \mathscr{F}.$$

This completes the proof. □

This completes the proof of Proposition 6.21. Here is the second proposition we need.

Proposition 6.24. *Let $\mathscr{F}, \mathscr{G} \in D^b(U)$. Then*

$$R\mathscr{H}om(\Psi_U(\mathscr{F}), r_U^{-1}\mathscr{G}[-1]) \simeq 0.$$

Proof. Let $V \subset U$ be an open subset. Then since $r_V^{-1}\mathscr{F} = r_V^!\mathscr{F}[-1]$, by adjunction we have

$$\begin{aligned} RHom(\Psi_U(\mathscr{F})|_V, r_U^{-1}\mathscr{G}[-1]|_V) &\simeq RHom(\Psi_V(\mathscr{F}|_V), r_V^{-1}\mathscr{G}[-1]|_V) \\ &\simeq RHom(Rr_{V,!}\Psi_V(\mathscr{F}|_V), \mathscr{G}[-2]|_V). \end{aligned}$$

Now we prove that $Rr_{V,!}\Psi_V(\mathscr{F}) \simeq 0$, $\forall\, \mathscr{F} \in D^b(V)$. Let $s' : M \times \mathbb{R}^2 \to M \times \mathbb{R}, (x, s, t) \mapsto (x, s+t)$ and $r' : M \times \mathbb{R}^2 \times \mathbb{R}_+ \to M \times \mathbb{R}^2, (x, s, t, u) \mapsto (x, s, t-u)$ (such that $s' \circ r' = r \circ s$). Then

$$\begin{aligned} Rr_{V,!}\Psi_V(\mathscr{F}) &\simeq Rr_{V,!}(Rs_!(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\Bbbk_\gamma))_{U_+} \\ &\simeq Rs'_!Rr'_!(\pi_1^{-1}\mathscr{F} \otimes \Bbbk_{\pi_2^{-1}\gamma} \otimes \Bbbk_{s^{-1}U_+}) \\ &\simeq Rs'_!Rr'_!((r')^{-1}(\pi_1')^{-1}\mathscr{F} \otimes \Bbbk_{\pi_2^{-1}\gamma \cap s^{-1}U_+}) \\ &\simeq Rs'_!((\pi_1')^{-1}\mathscr{F} \otimes Rr'_!\Bbbk_{\pi_2^{-1}\gamma \cap s^{-1}U_+}). \end{aligned}$$

(Here $\pi_1 : M \times \mathbb{R}^2 \times \mathbb{R}_+ \to M \times \mathbb{R}$ and $\pi_1' : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$.) Hence it suffices to show that $Rr'_!\Bbbk_{\pi_2^{-1}\gamma \cap s^{-1}U_+} \simeq 0$. This follows from direct calculation. □

Now by the previous two propositions and Lemma 6.17, we can deduce the Theorem 6.20. In fact, we make a remark here that using Lemma 6.23 and Proposition 6.24, one can show that our functor Ψ actually factors as

$$\begin{aligned} \Psi : \mu sh_\Lambda(\Lambda) &\xrightarrow{q^{-1}} \varinjlim_{\epsilon>0} \mu sh_{q_d q_\pi^{-1}(\Lambda)}(q_d q_\pi^{-1}(\Lambda) \cap \dot{T}^*(M \times \mathbb{R} \times (0,\epsilon))) \\ &\longleftarrow \varinjlim_{\epsilon>0} sh_{q_d q_\pi^{-1}(\Lambda) \cup r_d r_\pi^{-1}(\Lambda)}(M \times \mathbb{R} \times (0,\epsilon)) \end{aligned}$$

where the second functor is microlocalization along $q_d q_\pi^{-1}(\Lambda)$ (note that $q_d q_\pi^{-1}(\Lambda)$ is disjoint from $r_d r_\pi^{-1}(\Lambda)$).

6.4. **Quantization and Gluing.** Throughout this section we will assume that our conical Lagrangian $\Lambda \subset \dot{T}^*(M \times \mathbb{R})$ is the conification of an exact Lagrangian in T^*M, or equivalently, our Legendrian submanifold $\Lambda/\mathbb{R}_+$ is a Legendrian lift of the exact Lagrangian.

NWe are able to define the quantization of the conical Lagrangian $\Lambda \subset T^*(M \times \mathbb{R})$. Basically the construction is as follows. Let $\overline{\mathscr{F}} \in \mu sh_\Lambda(\Lambda_0)$ be represented by $\mathscr{F} \in D^b_{\Lambda_0}(U)$. Now we consider $\Psi_U(\mathscr{F}) \in D^b_{q_d q_\pi^{-1}(\Lambda_0) \cup r_d r_\pi^{-1}(\Lambda_0)}(M \times \mathbb{R} \times \mathbb{R}_+)$. Then

$$\mathscr{F}' = \Psi_U(\mathscr{F})|_{M \times \mathbb{R} \times \{\epsilon\}} \in D^b_{\Lambda_0 \cup T_\epsilon(\Lambda_0)}(M \times \mathbb{R})$$

where T_ϵ is the translation along $\mathbb{R}$ by ϵ. Finally we cut off the sheaf in an open subset $V \subset M \times \mathbb{R}$ diffeomorphic to $M \times \mathbb{R}$ such that $\Lambda_0 \subset V$ while $T_\epsilon(\Lambda_0) \cap V = \emptyset$. Then $\mathscr{F}|_V$ gives the sheaf we are seeking for.

However, one may note that this procedure we just described is completely cheating since we're starting from a globally defined sheaf and trying to construct such a sheaf (as if we don't know it is already there). The real problem is about gluing. Namely, we don't know (1). if the transition functions for sheaves on $D^b_{q_d q_\pi^{-1}(\Lambda_0) \cup r_d r_\pi^{-1}(\Lambda_0)}(M \times \mathbb{R} \times \mathbb{R}_+)$ gives transition functions on each slice; (2). if the translation T_ϵ doesn't change the morphism space between local representatives. This will be our task.

Here is where we need the condition that the Legendrian submanifold comes from an exact Lagrangian, which ensures that there are no Reeb chords and thus the sheaf theory on different slices doesn't change.

Definition 6.13. *For $u \in \mathbb{R}$ the translation T_u is $T_u : T^*(M\times\mathbb{R}) \to T^*(M\times\mathbb{R})$, $(x,\xi;t,\tau) \mapsto (x,\xi;t+u,\tau)$. For $\Lambda \subset T^*(M \times \mathbb{R})$, $\Lambda_u = \Lambda \cup T_u(\Lambda)$.*

Lemma 6.25. *Let $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$ be a conification of an exact Lagrangian in T^*M. Then there is a Hamiltonian isotopy $\varphi : \dot{T}^*(M \times \mathbb{R}) \times \mathbb{R}_+ \to \dot{T}^*(M \times \mathbb{R})$ such that*

$$d\varphi(\Lambda_1) = q_d q_\pi^{-1}(\Lambda) \cup r_d r_\pi^{-1}(\Lambda),\ \ \varphi_u(\Lambda_1) = \Lambda_u.$$

Let $\Lambda_+ = q_d q_\pi^{-1}(\Lambda) \cup r_d r_\pi^{-1}(\Lambda)$. Then by Guillermou-Kashiwara-Schapira, there are equivalences of categories

$$D^{lb}_{\Lambda_+}(M \times \mathbb{R} \times \mathbb{R}_+) \xrightarrow{\sim} D^{lb}_{\Lambda_+}(M \times \mathbb{R} \times (0,u)) \xrightarrow{\sim} D^{lb}_{\Lambda_{u'}}(M \times \mathbb{R}).$$

Corollary 6.26. *Let $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$ be a conification of an exact Lagrangian in T^*M. $\mathscr{F}, \mathscr{G} \in D^b_\Lambda(M \times \mathbb{R})$. Then*

$$RHom(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} RHom(q^{-1}\mathscr{F}, r^{-1}\mathscr{G}) \xrightarrow{\sim} RHom(\mathscr{F}, T_{u,*}\mathscr{G}).$$

Proof. Since r is a submersion with contractible fibers, we have

$$RHom(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} RHom(r^{-1}\mathscr{F}, r^{-1}\mathscr{G}).$$

Consider the exact triangle in Lemma 6.17 we have

$$RHom(r^{-1}\mathscr{F}, r^{-1}\mathscr{G}) \to RHom(q^{-1}\mathscr{F}, r^{-1}\mathscr{G}) \to RHom(\Psi_{M\times\mathbb{R}}(\mathscr{F}), r^{-1}\mathscr{G}) \xrightarrow{[1]}.$$

Since we know that $Rr_!\Psi_{M\times\mathbb{R}}(\mathscr{F}) \simeq 0$, the corollary is true. ∎

Now we start gluing. First let's introduce some notations.

Definition 6.14. *Let $\Lambda_0 \subset \Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$, $\{U_i\}_{i\in I}$ be an open covering of M. Denote by Λ_{ij} the connected components of $\Lambda \cap T^*U_i$. Let*

$$W_{ij} = \pi_1(\dot{\pi}_{U_i}(\Lambda_0 \cap \Lambda_{ij})),\ \ V_{ij} = U_i \cap (W_{ij} \times \mathbb{R}).$$

We assume that
(1). $\partial V_{ij} \cap U_i$ is smooth;
(2). $\dot{T}^*U_i \cap \Lambda_{ij} \cap (-N^*V_{ij}) = \emptyset$, $\dot{T}^*_{\Lambda_{ij}}\dot{T}^*U_i \cap T^*_{\partial \dot{T}^*V_{ij}}\dot{T}^*U_i = \emptyset$;
(3). $\dot{T}^*U_i \cap (\Lambda_{ij} + N^*V_{ij}) \cap (\Lambda_{ij'} + N^*V_{ij'}) = \emptyset$, $\forall\, j \neq j'$.
Under this assumption, we write

$$\Lambda_0^{\mathrm{ext}} = \bigcup_{i\in I, j\in J_i} (\dot{T}^*U_i \cap (\Lambda_{ij} + N^*V_{ij})).$$

Note that near a point, V_{ij} only depends on Λ_0, so Λ_0^{ext} only depends on Λ_0. In addition by (3) we know that $\Lambda_0^{\mathrm{ext}} \cap \Lambda = \bar{\Lambda}_0$.

Lemma 6.27. *Let $\Lambda \subset \dot{T}^*M$ be a conical Lagrangian and $\mathscr{F}, \mathscr{G} \in D^b(M)$ such that $\dot{SS}(\mathscr{F}) = \dot{SS}(\mathscr{G}) = \Lambda$. Let $V \subset M$ be open such that ∂V is smooth, and assume that*
(1). $(-N^*U) \cap \Lambda = \emptyset$;
(2). $\dot{T}^*_\Lambda \dot{T}^*M \cap \dot{T}^*_{\partial \dot{T}^*U}\dot{T}^*M = \emptyset$.
Then $\mu hom(R\Gamma_U(\mathscr{F}), \mathscr{G})|_{\dot{T}^*M} \simeq \mu hom(\mathscr{F}, \mathscr{G})_{\dot{T}^*U}|_{\dot{T}^*M}$.

Proof. Since $(-N^*U)\cap\Lambda = \emptyset$, $R\Gamma_V(\mathscr{F}) \simeq \mathscr{F}_{\overline{U}}$. Thus we have a natural morphism $\mathscr{F}_{\overline{U}} \to \mathscr{F}$, inducing

$$u : \mu hom(\mathscr{F}, \mathscr{G}) \to \mu hom(\mathscr{F}_{\overline{U}}, \mathscr{G}).$$

One can check that $\mathrm{supp}(\mu hom(\mathscr{F}_{\overline{U}}, \mathscr{G})) \subset \overline{T^*U}$, and u is an isomorphism on $\dot{T}^*U$. Now we check that

$$\dot{SS}(\mu hom(\mathscr{F}_{\overline{U}}, \mathscr{G})) \cap N^*(\dot{T}^*U) = \emptyset.$$

We know that $SS(\mu hom(\mathscr{F}_{\overline{U}}, \mathscr{G})) \subset C(\Lambda, \Lambda - N^*U)$. One can check this fact by hand.

Let $\mathscr{H}$ be the mapping cone of

$$u : \mu hom(\mathscr{F}, \mathscr{G})_{\dot{T}^*U} \to \mu hom(\mathscr{F}_{\overline{U}}, \mathscr{G}).$$

Then $\mathrm{supp}(\mathscr{H}) \subset \partial(\dot{T}^*U)$. Because the inclusion $\partial(\dot{T}^*U) \to T^*M$ is proper,

$$T^*_{\partial(\dot{T}^*U)}T^*M|_{\mathrm{supp}(\mathscr{H})} \subset SS(\mathscr{H}).$$

However, $\dot{SS}(\mathscr{H}) \cap N^*(\dot{T}^*U) = \emptyset$. Therefore $\mathrm{supp}(\mathscr{H})$ has to be empty. This finishes the proof. □

Proposition 6.28. *Let $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$ be the conification of an exact Lagrangian satisfying the assumptions. Let $c = (c_{jj'})_{j,j'\in\bigcup_i J_i}$ be a Cech coboundary, $(b_j)_{j\in\bigcup_i J_i}$ be its primitive, and $\overline{\mathscr{F}} \in (\mu sh_\Lambda(\Lambda_0))_c$ be a pure object with representative*

$$\mathscr{F}_{ij}\,(i \in I, j \in J_i),\ u_{ij,i'j'} \in H^{c_{jj'}}(\Lambda_0 \cap \Lambda_{jj'}, \mu hom(\mathscr{F}_{ij}, \mathscr{F}_{i'j'}))\,(i, i' \in I, j \in J_i, j' \in J_{i'}).$$

Then there exists $\epsilon > 0$, $\mathscr{F} \in D^b_{q_d q_\pi^{-1}(\Lambda)}(M \times \mathbb{R} \times (0, \epsilon))$ and isomorphisms

$$\varphi_i : \mathscr{F}|_{U_{i,\epsilon}} \xrightarrow{\sim} \bigoplus_{j\in J_i} \Psi_{U_i}\left(R\Gamma_{V_j}\mathscr{F}_j[b_j]\right)|_{U_{i,\epsilon}},$$

where $U_{i,\epsilon} = U_{i,+} \cap (M \times \mathbb{R} \times (0, \epsilon))$, such that

(1). $\mathrm{supp}(\mathscr{F}) \subset \overline{\gamma} \star \dot{\pi}_{M\times\mathbb{R}}(\overline{\Lambda}_0)$ *and*

$$\dot{SS}(\mathscr{F}) \subset q_d q_\pi^{-1}(\lambda_0^{ext}) \cup r_d r_\pi^{-1}(\lambda_0^{ext}) \cup T^*M \times (\mathbb{R} \times \mathbb{R}_+);$$

(2). $\varphi_i \circ \varphi_{i'}^{-1}|_{U_{ii'}}$ represents $u_{ij,i'j'} \in H^{c_{jj'}}(\Lambda_0 \cap \Lambda_{jj'}, \mu hom(\mathscr{F}_{ij}, \mathscr{F}_{i'j'}))$, $j \in J_i, j' \in J_{i'}$;

(3). φ_i induces an isomorphism $\mathscr{F} \xrightarrow{\sim} q^{-1}(\overline{\mathscr{F}})$.

Proof. Let U_i' be a neighbourhood of $\overline{U}_i$ on which $\mathscr{F}_i$ is defined, and Λ is non-characteristic for $\partial U_i'$. This will allow us to write down the isomorphisms

$$(\Bbbk_\Lambda)_{\dot{T}^*\overline{U}_i} \simeq R\Gamma_{\dot{T}^*U_i}\Bbbk_\Lambda,\ (\Bbbk_\Lambda)_{\dot{T}^*\overline{V}_i} \simeq R\Gamma_{\dot{T}^*V_i}\Bbbk_\Lambda.$$

By the assumption, we know that when $\Lambda_{ij} \cap \Lambda_{i'j'} = \emptyset$, we have

$$\begin{aligned}\mathrm{supp}(\mu hom((R\Gamma_{V_{ij}}(\mathscr{F}_i))_{V_{i'j'}}, \mathscr{F}_{i'})) &\subset \dot{SS}((R\Gamma_{V_{ij}}(\mathscr{F}_i))_{V_{i'j'}}) \cap \dot{SS}(\mathscr{F}_{i'}) \\ &\subset (\Lambda_{ij} + \dot{T}^*_{\partial V_{ij}}U_i + \dot{T}^*_{\partial V_{i'j'}}U_i) \cap \Lambda_{i'j'} = \emptyset.\end{aligned}$$

Hence by Lemma 6.27 one can obtain that

$$\begin{aligned}\mu hom((R\Gamma_{V_{ij}}(\mathscr{F}_i))_{V_{i'j'}}, \mathscr{F}_{i'})|_{\dot{T}^*\overline{U_{ii'}}} &\simeq \mu hom((\mathscr{F}_i)_{V_{i'j'}}, \mathscr{F}_{i'})|_{\dot{T}^*\overline{U}_{ii'}} \\ &\simeq \mu hom(\mathscr{F}_i, \mathscr{F}_{i'})_{\dot{T}^*\overline{V}_{ij}}|_{\dot{T}^*\overline{U}_{ii'}}.\end{aligned}$$

Now by the theorem in the previous section we have

$$\begin{aligned} H^k\left(U_{ii'}, R\mathscr{H}om(\mathscr{G}_i,\mathscr{G}_{i'})\right) &\simeq \bigoplus_{j\in J_i, j'\in J_{i'}} H^l\left(\dot{T}^*\overline{U}_{ii'}, \mu hom(R\Gamma_{V_{ij}}(\mathscr{F}_i), R\Gamma_{V_{i'j'}}(\mathscr{F}_j))\right) \\ &\simeq \bigoplus_{j\in J_i, j'\in J_{i'}} H^l\left(\dot{T}^*\overline{U}_{ii'}, \mu hom((R\Gamma_{V_{ij}}(\mathscr{F}_i))_{V_{i'j'}}, \mathscr{F}_{i'})\right) \\ &\simeq \bigoplus_{j\in J_i, j'\in J_{i'}} H^l\left(\dot{T}^*\overline{U}_{ii'}, \mu hom(\mathscr{F}_i, \mathscr{F}_{i'})|_{\dot{T}^*\overline{V}_{ij}}\right) \\ &\simeq \bigoplus_{j\in J_i, j'\in J_{i'}} H^l\left(\dot{T}^*\overline{U}_{ii'}\cap\Lambda_{ij}, \mu hom(\mathscr{F}_i, \mathscr{F}_{i'})\right). \end{aligned}$$

This completes the gluing procedure modulo technical issues about gluing sheaves by local sections. □

Proof of the main theorem. By the previous proposition there is a sheaf

$$\mathscr{F}_1 \in D^b_{\Lambda_+\cap\dot{T}^*(M\times\mathbb{R}\times(0,\epsilon))}(M\times\mathbb{R}\times(0,\epsilon)).$$

Since the sheaf category is invariant under Hamiltonian isotopies, there is a sheaf $\mathscr{F}_2 \in D^b_{\Lambda_+}(M\times\mathbb{R}\times\mathbb{R}_+)$ that extends $\mathscr{F}_1$. Suppose $\Lambda\subset\dot{T}^*(M\times[a,b])$. Then we restrict $\mathscr{F}_2$ to $M\times\mathbb{R}\times\{b-a+2\}$. Then Λ and $T_{b-a+2}(\Lambda)$ are separated by the hyperplane $t=b+1$. Consider a diffeomorphism

$$f: M\times\mathbb{R}\to M\times(-\infty,b+1),\ \ f|_{M\times(-\infty,b]}=\mathrm{id}.$$

Let $\mathscr{F}=f^{-1}(\mathscr{F}_2|_{M\times(-\infty,b+1)})$. Then $\mathscr{F}$ satisfies the condition we need. □

6.5. **Behaviour of the Sheaf.** Before proving any further results using the existence of sheaf quantization, we first explore the behaviour of the sheaf, and in particular the behaviour at infinity, in this section.

Definition 6.15. *Let $\Lambda\subset T^*(M\times\mathbb{R})$ be the conification of a compact Lagrangian $L\subset T^*M$, in particular $\pi(\Lambda)\subset M\times[-A,A]$. Let $\mathscr{F}\in D^b_\Lambda(M\times\mathbb{R})$. Then*

$$\mathscr{F}_-=\mathscr{F}|_{M\times\{-2A\}},\ \ \mathscr{F}_+=\mathscr{F}|_{M\times\{2A\}}.$$

The category $D^b_{\Lambda,0}(M\times\mathbb{R})$ is the full subcategory of $D^b_\Lambda(M\times\mathbb{R})$ consisting of sheaves with $\mathscr{F}_-\simeq 0$.

Proposition 6.29. *Let $\mathscr{F},\mathscr{G}\in D^b_{\Lambda,0}(M\times\mathbb{R})$. Then*

$$RHom(\mathscr{F},\mathscr{G})\simeq RHom(\mathscr{F}_+,\mathscr{G}_+)\simeq R\Gamma(\Lambda,\mu hom(\mathscr{F},\mathscr{G})).$$

Proof. The isomorphism $RHom(\mathscr{F},\mathscr{G})\simeq R\Gamma(\Lambda,\mu hom(\mathscr{F},\mathscr{G}))$ is essentially what we have proved. Thus it suffices to show that $RHom(\mathscr{F},\mathscr{G})\simeq RHom(\mathscr{F}_+,\mathscr{G}_+)$. We apply Corollary 6.26, for $u\geq 2A$,

$$\begin{aligned} RHom(\mathscr{F},\mathscr{G}) &\simeq RHom(\mathscr{F},T_{u,*}\mathscr{G})\simeq RHom(\pi_M^{-1}\mathscr{F}_+,T_{u,*}\mathscr{G}) \\ &\simeq RHom(\mathscr{F}_+,\pi_{M,*}T_{u,*}\mathscr{G}). \end{aligned}$$

Consider the exact triangle

$$T_{u,*}\mathscr{G}\otimes\Bbbk_{(-\infty,A+u)}\to T_{u,*}\mathscr{G}\to\pi_M^{-1}\mathscr{G}_+\otimes\Bbbk_{[A+u,+\infty)}\xrightarrow{[1]}.$$

Now we show that $\pi_{M,*}(T_{u,*}\mathscr{G}\otimes\Bbbk_{(-\infty,A+u)})\simeq 0$. Since $SS(\mathscr{G})\subset T^*_{\geq 0}(M\times\mathbb{R})$ and $\mathrm{supp}(T_{u,*}\mathscr{G}\otimes\Bbbk_{(-\infty,A+u)})$ is compact, by microlocal Morse lemma

$$\pi_{M,*}(T_{u,*}\mathscr{G}\otimes\Bbbk_{(-\infty,A+u)})\simeq\pi_{M,*}(T_{u,*}\mathscr{G}\otimes\Bbbk_{(-\infty,A+u)}|_{(-\infty,-A)})\simeq 0.$$

Hence we are through. □

Proposition 6.30. *Let $\Bbbk = \mathbb{Z}$ or a finite field, and $\mathscr{F} \in D^b_{\Lambda,0}(M \times \mathbb{R})$ be simple. Then $\mathscr{F}_+$ is concentrated in a single degree and has rank 1.*

Proof. First let $\Bbbk$ be a finite field and prove that $\mathscr{F}_+$ is concentrated in a single degree. Since $|\Bbbk| < \infty$, one can choose a finite cover $r : \tilde{M} \to M$ such that $r^{-1}\mathscr{F}_+$ is a trivial local system. Write $r' = r \times \mathrm{id}_{\mathbb{R}}$, $\Lambda' = (dr')^{-1}(\Lambda)$ and $\mathscr{F}' = (r')^{-1}\mathscr{F}$. Then by finiteness of r, compactness is preserved,

$$RHom(\mathscr{F}'_+, \mathscr{F}'_+) \simeq R\Gamma(\Lambda', \mu hom(\mathscr{F}', \mathscr{F}')) = R\Gamma(\Lambda', \Bbbk_{\Lambda'}).$$

The left hand side is symmetrically indexed, while the right hand side is concentrated in nonnegative degrees. Thus both sides are concentrated in degree 0, which is just what we claim.

We show that $\mathscr{F}_+$ is of rank 1. By the isomorphism we used, it suffices to show that Λ' has only 1 component. Suppose the stalk of $\mathscr{F}_+$ is $\Bbbk^d$. Then the dimension of the left hand side is d^2 and can have at most d independent idempotents. However, we know that the number of independent idempotents is equal to the number of connected components. Therefore $d = 1$.

Finally let $\Bbbk = \mathbb{Z}$. First suppose the stalk of $\mathscr{F}_+$ has p-torsion, then by universal coefficient theorem we know that $\mathscr{F}_+ \otimes \mathbb{Z}/p\mathbb{Z}$ cannot be concentrated in a single degree. A contradition. Since $\mathscr{F}_+$ is free and its localization along any prime p is concentrated in a single degree and has rank 1, the claim is proved. □

6.6. **Topological Consequences.** In this section we show how the sheaf quantization result can be used to show the restrictions on exact Lagrangians in cotangent bundles. The nearby Lagrangian conjecture predicts that ALL closed exact Lagrangians L in a cotangent bundle (of a closed manifold) T^*Q are Hamiltonian isotopic to the zero section. Currently, the result by Abouzaid and Kragh claims that the projection

$$\pi : L \to M$$

induces a (simple) homotopy equivalence. Now we prove the result.

Theorem 6.31. *Let L be a closed exact Lagrangian submanifold in T^*M where M is closed. Then the Maslov class $\mu_\Lambda = 0$ and $\pi : L \to M$ induces a homotopy equivalence.*

We first prove homotopy equivalence under the assumption of vanishing of Maslov class and the relative Stiefel-Whitney class.

Proposition 6.32. *Let Λ be the conification of a compact exact Lagrangian $L \subset T^*M$ where M. Suppose $\mu_\Lambda = rw_{2,\Lambda} = 0$. Then the natural projection $\pi : \Lambda \to M$ induces an isomorphism $\pi^* : H^*(M; \Bbbk_M) \xrightarrow{\sim} H^*(\Lambda; \Bbbk_\Lambda)$.*

Proof. Choose a simple object $\mathscr{F} \in \mu sh_\Lambda(\Lambda)$ with preimage $\mathscr{F} \in D^b_{\Lambda,0}(M \times \mathbb{R})$. Let $\Bbbk = \mathbb{Z}$. Then since by Proposition 6.30 $\mathscr{F}_+$ is a rank 1 local system on M,

$$RHom(\mathscr{F}_+, \mathscr{F}_+) \simeq R\Gamma(M; \Bbbk_M).$$

On the other hand, by Proposition 6.29 we have

$$RHom(\mathscr{F}_+, \mathscr{F}_+) \simeq R\Gamma(\Lambda; \mu hom(\mathscr{F}, \mathscr{F})) = R\Gamma(\Lambda; \Bbbk_\Lambda).$$

Finally one need to check that this map is indeed induced by the projection (which is not so hard). This proves the proposition. □

We then prove the vanishing result for the Maslov class of the exact Lagrangian.

Theorem 6.33. *Let $\Lambda \subset T^*(M\times\mathbb{R})$ be the conification of a compact Lagrangian $L \subset T^*M$. Then $\mu_L = \mu_\Lambda = 0$.*

One may want to start with a global section on $\mu sh_\Lambda(\Lambda)$. First we need the following lemma, which will allow us to lift Λ as well as M simultaneously to a cyclic cover $\tilde{\Lambda}$ and $\tilde{M}$ so that $\mu_{\tilde{\Lambda}} = 0$.

Lemma 6.34. *The natural projection $\pi : \Lambda \to M$ induces an injection $\pi_* : \pi_1(\Lambda) \to \pi_1(M)$.*

Proof. Let $\mathscr{L}$ be the local system on Λ corresponding to the regular representation of $\pi_1(\Lambda)$. There exists a simple object $\mathscr{F}_0 \in D^b_{/[1],\Lambda}(M\times\mathbb{R})$, where $D^b_{/[1]}(M\times\mathbb{R})$ is the sheaf of categories localized along the shifting functor [1]. Now there exists a unique $\mathscr{F}_1 \in D^b_{/[1],\Lambda}(M\times\mathbb{R})$ such that

$$\mu hom(\mathscr{F}_0, \mathscr{F}_1) \simeq \mathscr{L}.$$

Using Proposition 6.29, one can find that in fact

$$\pi^{-1}\mathscr{F}_{1,+} \simeq \mathscr{L} \otimes \pi^{-1}\mathscr{F}_{0,+}.$$

Let $\mathscr{L}_i$ be the sheaf associated to $U \mapsto Hom(\mathbb{Z}/2\mathbb{Z}_U, \mathscr{F}_{i,+})$. These local systems correspond to representations $\rho_{0,1}$ of $\pi_1(M)$, and induce representations of $\pi_1(\Lambda)$. In addition we have $\rho_1 \simeq \rho_{\mathrm{reg}} \otimes \rho_0$. $\rho_i|_{\ker(\pi_*)}$ are trivial, so is $\rho_{\mathrm{reg}}|_{\ker(\pi_*)}$. This shows that $\ker(\pi_*) = 1$. ☐

Proof of Theorem 6.33. We view μ_Λ as a map $\pi_1(\Lambda) \to \mathbb{Z}$. Consider the diagram

$$\begin{array}{ccc} \tilde{\Lambda} & \xrightarrow{df} & \Lambda \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{f} & M \end{array}$$

where $f : \tilde{M} \to M$ is the universal cover, and $\tilde{\Lambda}$ is a connected component of $df^{-1}(\Lambda)$. By the lemma, we know that $\tilde{\Lambda}$ is actually the universal cover of Λ. Now we in addition consider

$$\begin{array}{ccccc} \tilde{\Lambda} & \longrightarrow & \tilde{\Lambda}/\ker(\mu_\Lambda) & \longrightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & \tilde{M}/\ker(\mu_\Lambda) & \longrightarrow & M. \end{array}$$

When $\mu_\Lambda \neq 0$ the right half square is a diagram of cyclic covering.

Since $\mu_{\tilde{\Lambda}} = 0$ one can pick an object $\mathscr{F} \in \mu sh_{\tilde{\Lambda}}(\tilde{\Lambda})$. Then a deck transformation φ on $\tilde{M}$ will give $\varphi^{-1}\mathscr{F} \simeq \mathscr{F}[m_\Lambda]$ where m_Λ is the Maslov number. This tells us that $\mathscr{F}_+ \simeq \mathscr{F}_+[m_\Lambda]$. However by Proposition 6.29, since $\mathscr{F}$ is bounded, $\mathscr{F}_+$ must also be bounded. A contradiction. ☐

Proposition 6.35. *Let $\Lambda \subset T^*(M\times\mathbb{R})$ be the conification of a compact Lagrangian $L \subset T^*M$. Then $rw_{2,L} = rw_{2,\Lambda} = 0$.*

Proof. Note that $(\mathbb{Z}/2\mathbb{Z})^\times$ is trivial, so when $\Bbbk = \mathbb{Z}/2\mathbb{Z}$, $\mu sh_\Lambda(\Lambda)$ always has a simple global object. By Proposition 6.32,

$$\pi^* : H^2(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^2(\Lambda;\mathbb{Z}/2\mathbb{Z}).$$

Hence $rw_{2,L} = rw_{2,M} = 0$. ☐

These two propositions above tell us that $R\Gamma(\Lambda;\Bbbk) \simeq R\Gamma(M;\Bbbk)$ without any further assumptions.

Proposition 6.36. *The natural projection $\pi : \Lambda \to M$ induces an isomorphism $\pi_* : \pi_1(\Lambda) \xrightarrow{\sim} \pi_1(M)$.*

Proof. We show that first $\pi_1^{-1} : Loc(M) \xrightarrow{\sim} Loc(\Lambda)$ is an equivalence and second $R\Gamma(M, \mathscr{L}) \simeq R\Gamma(\Lambda, \pi^{-1}\mathscr{L})$. Let $\mathscr{F} \in D^b_{\Lambda,0}(M \times \mathbb{R})$ be a simple sheaf.

On the one hand, we have

$$\begin{aligned} RHom(\mathscr{L}, \mathscr{L}') &\simeq RHom(\mathscr{F}_+ \otimes \mathscr{L}, \mathscr{F} \otimes \mathscr{L}') \\ &\simeq R\Gamma(\Lambda, \mu hom(\mathscr{F} \otimes \pi_M^{-1}\mathscr{L}, \mathscr{F} \otimes \pi_M^{-1}\mathscr{L}')) \\ &\simeq R\Gamma(\Lambda, \mu hom(\mathscr{F}, \mathscr{F}) \otimes R\mathscr{H}om(\pi^{-1}\mathscr{L}, \pi^{-1}\mathscr{L}')) \\ &\simeq RHom(\pi^{-1}\mathscr{L}, \pi^{-1}\mathscr{L}'). \end{aligned}$$

On the other hand, note that $\mu hom(\mathscr{F}, -)$ induces an equivalence between $\mu sh_\Lambda(\Lambda)$ and $D^bLoc(\Lambda)$, so for any $\mathscr{L}_\Lambda \in Loc(\Lambda)$ there is a preimage $\mathscr{G} \in \mu sh_\Lambda(\Lambda)$. This will give us $\mathscr{G} \in D^b_{\Lambda,0}(M \times \mathbb{R})$ and

$$\mu hom(\mathscr{F}, \mathscr{G})|_\Lambda \simeq \mathscr{L}_\Lambda.$$

Without loss of generality we may also assume that $\mathscr{F}_+ \simeq \Bbbk_M$. Then

$$\mathscr{L}_M = \mathscr{G}_+ \simeq (\mathscr{F} \otimes \pi^{-1}\mathscr{L}_M)_+.$$

By Proposition 6.29 this means $\mathscr{G} \simeq \mathscr{F} \otimes \pi^{-1}\mathscr{L}'$. Hence

$$\mathscr{L}_\Lambda \simeq \mu hom(\mathscr{F}, \mathscr{G})|_\Lambda \simeq R\mathscr{H}om(\mathscr{F}_+, \mathscr{G}_+) = \pi^{-1}\mathscr{L}_M.$$

This proves the equivalence.

Finally, to check that $R\Gamma(M, \mathscr{L}_M) \simeq R\Gamma(\Lambda, \pi^{-1}\mathscr{L}_M)$, it suffices to use the fact that $\mathscr{G} \simeq \mathscr{F} \otimes \pi^{-1}\mathscr{L}_M$ and Proposition 6.29. This finally shows the isomorphism on fundamental groups $\pi_* : \pi_1(\Lambda) \to \pi_1(M)$. $\square$

Now the main theorem follows from Whitehead theorem.